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Vector coherent state theory of the non-compact orthosymplectic superalgebras: I. General theory

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Abstract. Vector coherent states are defined for the positive discrete series irreducible representations of the non-compact orthosymplectic superalgebras $\mathrm{osp}(P/2N, \mathbb{R})$, where $P=2M$ or $2M+1$. An orthonormal Bargmann-Berezin basis, symmetry-adapted to $\mathrm{osp}(P/2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus \mathrm{u}(N)$, is constructed and used to develop the K-matrix theory for $\mathrm{osp}(P/2N, \mathbb{R})$. A general method is provided for determining the conditions of existence of star representations (and of grade star representations in the $\mathrm{osp}(2/2N, \mathbb{R})$ case), and the branching rule for their decomposition into a direct sum of $\mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R})$ irreducible representations. As a by-product, it also enables the matrix elements of the odd generators between basis states of lowest-weight $\mathrm{so}(P) \oplus \mathrm{u}(N)$ irreducible representations to be calculated in a straightforward way.

1. Introduction

Since their introduction (Deenen and Quesne 1984a, Rowe 1984), the vector coherent states (VCS), also called partially coherent states, have come to play an increasingly important role in Lie algebra representation theory. Their combination with K-matrix theory indeed provides a simple systematic procedure for explicitly constructing the ladder irreducible representations (irreps) of an algebra g in bases symmetry-adapted to some maximal-rank subalgebra g_0 (Hecht 1987, Rowe *et al* 1988).

In standard (generalized) coherent state (CS) theory (Perelomov 1972, 1977, Gilmore 1972, 1974), the irreps of a group G are induced from the one-dimensional irreps of a subgroup G_0. VCS theory arises as a natural extension of the latter when finite-dimensional vector irreps of G_0 are considered instead of one-dimensional irreps (Rowe *et al* 1985a). VCS theory is also intimately connected with the Lie algebra contraction-expansion procedures and with boson representations (Rosensteel and Rowe 1981, Deenen and Quesne 1985, Quesne 1987).

In K-matrix theory, the difficult determination of the VCS identity resolution integral form is replaced by an implicit definition of the VCS scalar product (Rowe 1984, Rowe *et al* 1984, 1988, Deenen and Quesne 1982, 1984b, 1985, Castaños *et al* 1985, Hecht 1987). This is achieved by specifying an orthonormal basis with respect to such a scalar product. For this purpose, one starts from a basis of vector-valued functions, orthonormal with respect to a Bargmann scalar product (Bargmann 1961), and hence connected with standard boson CS theory (Glauber 1963a, b); then one maps the vector Bargmann

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basis onto an orthonormal VCS basis by means of a transformation K. In determining the K matrix, full use is made of tensor calculus with respect to the subalgebra g_0. Finally, the explicit matrices of the generators of g are directly given in terms of the K-matrix elements.

The basic role played by Lie algebra gradings in the VCS construction has been recognized recently (Rowe *et al* 1988, Le Blanc and Rowe 1988), thereby making possible its extension to some non-semisimple Lie algebras (Quesne 1990b). Such gradings, which are fundamental to the Tits-Koecher-Kantor construction of finite-dimensional simple Lie algebras, can be generalized to Lie superalgebras to give a unified construction of both types of mathematical structures by using ternary algebras as building blocks (Bars and Günaydin 1979). It is therefore obvious that the VCS and K-matrix combined theory can be extended to Lie superalgebras without substantial modification (Le Blanc and Rowe 1989, 1990). The only significant change consists in replacing the Bargmann scalar product by a Bargmann-Berezin one (Berezin 1966), connected with boson-fermion CS theory (Ohnuki and Kashiwa 1978), because one now has to deal with functions depending on both complex and Grassmann variables.

The purpose of the present series of papers is to develop the VCS theory for the positive discrete series irreps of the non-compact orthosymplectic superalgebras $\mathrm{osp}(P/2N, \mathbb{R})$ (where $P=2M$ or $2M+1$) in $\mathrm{osp}(P/2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus u(N)$ bases, recently presented in a preliminary account (Quesne 1990a). This new application of VCS theory is of considerable practical interest because non-compact orthosymplectic superalgebras make their appearance in applications of supersymmetry to a lot of physical problems (see, e.g. de Crombrugghe and Rittenberg 1983, Wegner 1983, Verbaarschot *et al* 1985, Günaydin and Warner 1986, Schmitt *et al* 1988, 1989). With the recent study of the highest-weight finite-dimensional irreps for the compact form of the orthosymplectic superalgebras (Le Blanc and Rowe 1990), the present work completes the review of the most important representations of this class of classical superalgebras.

In this paper we present the general theory valid for all $\mathrm{osp}(P/2N, \mathbb{R})$ superalgebras including the $\mathrm{osp}(2/2N, \mathbb{R})$ ones, which need special treatment. In the following paper (henceforth referred to as II) (Quesne 1990d), we shall illustrate the general theory with some detailed examples corresponding to the cases $P=1, 2, 3$ and 4.

Sections 2-6 of the present paper mostly deal with the $\mathrm{osp}(P/2N, \mathbb{R})$ superalgebras corresponding to $P \neq 2$. In section 2, the definition of the $\mathrm{osp}(P/2N, \mathbb{R})$ superalgebras and of their positive discrete series irreps is reviewed and the existence of a five-dimensional $\mathbb{Z}$-graded structure with respect to the maximal compact even subalgebra $\mathrm{so}(P) \oplus u(N)$ is established. In section 3, the $\mathrm{osp}(P/2N, \mathbb{R})$ VCS are introduced and shown to generalize the standard CS of the most degenerate positive discrete series irreps of $\mathrm{osp}(1/2N, \mathbb{R})$ and $\mathrm{osp}(2/2N, \mathbb{R})$, recently studied by Balantekin *et al* (1988, 1989). The VCS representation of the $\mathrm{osp}(P/2N, \mathbb{R})$ generators is also obtained. In section 4, an orthonormal vector Bargmann-Berezin basis, symmetry-adapted to $\mathrm{osp}(P/2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus u(N)$, is constructed and its relation with boson-fermion CS is stressed. This basis is then used in the K-matrix theory of $\mathrm{osp}(P/2N, \mathbb{R})$ developed in section 5. Section 6 contains some general results for the reduced matrix elements of $\mathrm{so}(P) \oplus \mathrm{u}(N)$ irreducible tensors, whose evaluation is required for implementing K-matrix theory in practical cases, as is done in II. Finally, section 7 emphasizes the differences occurring in the $\mathrm{osp}(2/2N, \mathbb{R})$ case with respect to the general theory presented for the $\mathrm{osp}(P/2N, \mathbb{R})$ superalgebras with $P \neq 2$ in the previous sections.

2. The osp($P/2N$, ℝ) superalgebras and their positive discrete series irreps

The non-compact osp($P/2N,\mathbb{R}$) superalgebra ($P=2M$ or $2M+1$) is spanned by the operators

$$\Lambda_{AB}=(-1)^{\eta_A\eta_B}\Lambda_{BA} \qquad A, B=(0),\pm 1,\ldots,\pm(M+N) \tag{2.1}$$

satisfying the supercommutation relations

$$[\Lambda_{AB},\Lambda_{CD}\}=g_{CB}\Lambda_{AD}-(-1)^{(\eta_A+\eta_B)(\eta_C+\eta_D)}g_{AD}\Lambda_{CB} + (-1)^{\eta_A\eta_B}[g_{CA}\Lambda_{BD}-(-1)^{(\eta_A+\eta_B)(\eta_C+\eta_D)}g_{BD}\Lambda_{CA}] \tag{2.2}$$

where

$$\eta_A=\begin{cases}1 & \text{if } A=(0),\pm 1,\ldots,\pm M \quad (2.3a)\\ 0 & \text{if } A=\pm(M+1),\ldots,\pm(M+N) \quad (2.3b)\end{cases}$$

$$g_{AB}=\delta_{A,-B}\varepsilon_A \tag{2.4}$$

and

$$\varepsilon_A=\begin{cases}1 & \text{if } A=(0),\pm 1,\ldots,\pm M \quad (2.5a)\\ A/|A| & \text{if } A=\pm(M+1),\ldots,\pm(M+N). \quad (2.5b)\end{cases}$$

The range of indices A, B in (2.1) and of index A in (2.3a) and (2.5a) includes 0 only for $P=2M+1$.

To ensure that the even part of the superalgebra is $g_{\bar 0}=\mathrm{so}(P)\oplus\mathrm{sp}(2N,\mathbb{R})$, one has to impose some adjoint conditions on the corresponding generators. We shall use

$$\begin{aligned}&A^{\dagger}_{ab}=-A^{\dagger}_{ba}=\Lambda_{ab} \qquad A^{ab}=-A^{ba}=(A^{\dagger}_{ab})^{\dagger}=\Lambda_{-b,-a}\\ &B^{\dagger}_{a}=\Lambda_{a0} \qquad B^{a}=(B^{\dagger}_{a})^{\dagger}=\Lambda_{0,-a} \qquad C_a{}^{b}=(C_b{}^{a})^{\dagger}=\Lambda_{a,-b}\\ &a,b=1,\ldots M\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&D^{\dagger}_{ij}=D^{\dagger}_{ji}=\Lambda_{M+i,M+j} \qquad D^{ij}=D^{ji}=(D^{\dagger}_{ij})^{\dagger}=\Lambda_{-M-i,-M-j}\\ &E_i{}^{j}=(E_j{}^{i})^{\dagger}=\Lambda_{M+i,-M-j} \qquad i,j=1,\ldots,N\end{aligned} \tag{2.7}$$

to denote the so(P) and sp($2N,\mathbb{R}$) generators, respectively. The operators $C_a{}^b$ and $E_i{}^j$ span the u(M) and u(N) subalgebras of so(P) and sp($2N,\mathbb{R}$), respectively. For the generators of the odd part $g_{\bar 1}$, we shall use the following notation:

$$\begin{aligned}&I_{ai}=\Lambda_{a,M+i} \qquad G^{ai}=\Lambda_{-a,-M-i} \qquad H_i{}^{a}=\Lambda_{-a,M+i} \qquad J_a{}^{i}=\Lambda_{a,-M-i}\\ &K_i=\Lambda_{0,M+i} \qquad F^{i}=\Lambda_{0,-M-i} \qquad a=1,\ldots,M \qquad i=1,\ldots,N.\end{aligned} \tag{2.8}$$

The operators $B^{\dagger}_a$, B^a and K_i, F^i, defined in (2.6) and (2.8), respectively, only exist for $P=2M+1$. The supercommutators of the operators (2.6)–(2.8) are listed in the appendix.

The adjoint operation in so(P)$\oplus$sp($2N,\mathbb{R}$) can be extended to an adjoint operation in osp($P/2N,\mathbb{R}$) in two ways differing by a choice of sign (Scheunert *et al* 1977):

$$(F^{i})^{\dagger}=F^{\dagger}_i=\pm K_i \qquad (G^{ai})^{\dagger}=G^{\dagger}_{ai}=\pm I_{ai} \qquad (J_a{}^{i})^{\dagger}=\pm H_i{}^{a}. \tag{2.9}$$

The adjoint conditions contained in (2.6), (2.7) and (2.9) may be written in a compact form as

$$\Lambda_{AB} = (\pm 1)^{\eta_A+\eta_B}(\Lambda_{-B,-A})^{\dagger}. \tag{2.10}$$

For $P=2$, and only in this case, the adjoint operation in the Lie subalgebra can also be extended to a grade adjoint operation in the superalgebra (Scheunert *et al* 1977). Hence, osp$(P/2N,\mathbb{R})$, $P\neq 2$, may have star but no grade star representations, while osp$(2/2N,\mathbb{R})$ may have star and grade star representations, as well as representations which are both star and grade star (in either case, representations which are neither star nor grade star exist, of course). Up to the end of section 6, we shall restrict ourselves to P values different from 2, so that equations (2.9) and (2.10) will be valid and we shall only deal with star representations.

When the upper signs are chosen in (2.9) and (2.10), the osp$(2M/2N,\mathbb{R})$ (respectively, osp$(2M+1/2N,\mathbb{R})$) generators can be realized in a super Fock space $\mathscr{F}$ (Günaydin 1988, Günaydin and Hyun 1988) as bilinear operators in Mn (respectively, $(M+1)n$) pairs of fermion creation and annihilation operators $a^{\dagger}_{as}$, a^{as}, $a=1,2,\ldots,M$, $s=1,2,\ldots,n$, (respectively, $a=1,2,\ldots,M+1$, $s=1,2,\ldots,n$), and Nn pairs of boson creation and annihilation operators $b^{\dagger}_{is}$, b^{is}, $i=1,2,\ldots,N$, $s=1,2,\ldots,n$, as follows:

$$\begin{aligned}
&A^{\dagger}_{ab}=\sum_s a^{\dagger}_{as}a^{\dagger}_{bs} \qquad A^{ab}=\sum_s a^{bs}a^{as} \qquad C_a{}^b=\sum_s a^{\dagger}_{as}a^{bs}-\tfrac{1}{2}n\delta_a{}^b\\
&B^{\dagger}_a=\frac{1}{\sqrt{2}}\sum_s a^{\dagger}_{as}(a^{\dagger}_{M+1,s}+a^{M+1,s}) \qquad B^a=\frac{1}{\sqrt{2}}\sum_s (a^{\dagger}_{M+1,s}+a^{M+1,s})a^{as}\\
&D^{\dagger}_{ij}=\sum_s b^{\dagger}_{is}b^{\dagger}_{js} \qquad D^{ij}=\sum_s b^{js}b^{is} \qquad E_i{}^j=\sum_s b^{\dagger}_{is}b^{js}+\tfrac{1}{2}n\delta_i{}^j\\
&G^{\dagger}_{ai}=\sum_s a^{\dagger}_{as}b^{\dagger}_{is} \qquad G^{ai}=\sum_s b^{is}a^{as} \qquad H_i^a=\sum_s b^{\dagger}_{is}a^{as} \qquad J_a{}^i=\sum_s a^{\dagger}_{as}b^{is}\\
&F^{\dagger}_i=\frac{1}{\sqrt{2}}\sum_s b^{\dagger}_{is}(a^{\dagger}_{M+1,s}+a^{M+1,s}) \qquad F^i=\frac{1}{\sqrt{2}}\sum_s (a^{\dagger}_{M+1,s}+a^{M+1,s})b^{is}.
\end{aligned} \tag{2.11}$$

Here all the summations run over the range $1,\ldots,n$, where n is an even integer, large enough to allow the most general positive discrete series irreps of osp$(P/2N,\mathbb{R})$, as defined below, to be realized in the super Fock space.

The weight generators of osp$(P/2N,\mathbb{R})$ are those of so$(P)\oplus$sp$(2N,\mathbb{R})$ or, equivalently, those of so$(P)\oplus$u(N). We choose to enumerate them in the order $E_1{}^1$, $E_2{}^2,\ldots,E_N{}^N$, $C_1{}^1$, $C_2{}^2,\ldots,C_M{}^M$. From the supercommutation relations (A.1)-(A.4), it is then clear that the lowering generators are A^{ab}, B^a, $C_a{}^b$ $(a>b)$, D^{ij}, $E_i{}^j$ $(i>j)$, F^i, G^{ai}, and $J_a{}^i$, whereas the raising generators are $A^{\dagger}_{ab}$, $B^{\dagger}_a$, $C_a{}^b$ $(a<b)$, $D^{\dagger}_{ij}$, $E_i{}^j$ $(i<j)$, K_i, I_{ai}, and H_i^a.

In most physical applications, one is interested in the decomposition of the osp$(P/2N,\mathbb{R})$ irreps into irreps of the Lie subalgebra so$(P)\oplus$sp$(2N,\mathbb{R})$. In the solution to this problem considered in the present series of papers, a central role is played by the maximal compact even subalgebra so$(P)\oplus$u(N), henceforth referred to as the stability subalgebra of osp$(P/2N,\mathbb{R})$. We shall therefore consider the chain

$$\text{osp}(P/2N,\mathbb{R})\supset\text{so}(P)\oplus\text{sp}(2N,\mathbb{R})\supset\text{so}(P)\oplus\text{u}(N). \tag{2.12}$$

This has to be contrasted with the chain osp$(P/2N,\mathbb{R})\supset$u(M/N), considered by Günaydin (1988) and Günaydin and Hyun (1988), where u(M/N) is the maximal compact subsuperalgebra of osp$(P/2N,\mathbb{R})$, generated by the operators $C_a{}^b$, E_i^j, H_i^a and $J_a{}^i$.

The VCS construction, to be carried out in section 3, will be based on the property that $g = \mathrm{osp}(P/2N, \mathbb{R})$ has a five-dimensional $\mathbb{Z}$-graded structure (Bars and Günaydin 1979) with respect to its maximal compact even subalgebra $\mathrm{so}(P) \oplus \mathrm{u}(N)$:

$$g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \tag{2.13}$$

where

$$\begin{aligned} &g_{-2} = \mathrm{span}\{D^{ij}\} \qquad g_{-1} = \mathrm{span}\{F^i, G^{ai}, J_a{}^i\} \qquad g_0 = \mathrm{so}(P) \oplus \mathrm{u}(N) \\ &g_1 = \mathrm{span}\{K_i, I_{ai}, H_i{}^a\} \qquad g_2 = \mathrm{span}\{D^\dagger_{ij}\}. \end{aligned} \tag{2.14}$$

There does indeed exist an operator $\hat{\mathcal{N}}$, belonging to the maximal-rank subalgebra g_0, such that

$$[\hat{\mathcal{N}}, g_\rho\} = \rho g_\rho \qquad \rho = -2, \ldots, 2. \tag{2.15}$$

In other words, the elements of $\mathrm{osp}(P/2N, \mathbb{R})$ belonging to various subspaces g_ρ satisfy the supercommutation relations

$$[g_\rho, g_\sigma\} \subset g_{\rho+\sigma} \qquad \rho, \sigma = -2, \ldots, 2. \tag{2.16}$$

The $\mathbb{Z}$-grading operator is

$$\hat{\mathcal{N}} = E_i{}^i \tag{2.17}$$

where from now on we assume that there is a summation over repeated covariant and contravariant indices.

We observe that the $\mathbb{Z}$ gradation is consistent with the $\mathbb{Z}_2$ gradation defining the superalgebra since

$$g_{\bar{0}} = \mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R}) = g_{-2} \oplus g_0 \oplus g_2 \tag{2.18a}$$

and

$$g_{\bar{1}} = g_{-1} \oplus g_1. \tag{2.18b}$$

In particular, equation (2.18*a*) shows that the intermediate algebra of (2.12) has a three-dimensional $\mathbb{Z}$-graded structure.

We shall consider here those star irreps of $\mathrm{osp}(P/2N, \mathbb{R})$ (and also in section 7 those grade star irreps of $\mathrm{osp}(2/2N, \mathbb{R})$) which can be induced from a lowest-weight $\mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R})$ irrep $[\Xi] \oplus \langle\Omega\rangle$ or, equivalently, from a lowest-weight $\mathrm{so}(P) \oplus \mathrm{u}(N)$ irrep $[\Xi] \oplus \{\Omega\}$. Here $[\Xi]$, $\langle\Omega\rangle$ and $\{\Omega\}$ are shorthand notations for $[\Xi_1\Xi_2 \ldots \Xi_M]$, $\langle\Omega_1\Omega_2 \ldots \Omega_N\rangle$ and $\{\Omega_1\Omega_2 \ldots \Omega_N\}$, where $\Xi_1, \ldots, \Xi_M$, $\Omega_1, \ldots, \Omega_N$ are some integers subject to the conditions $\Xi_1 \geqslant \Xi_2 \geqslant \ldots \geqslant \Xi_{M-1} \geqslant |\Xi_M|$ or $\Xi_1 \geqslant \Xi_2 \geqslant \ldots \geqslant \Xi_M \geqslant 0$ according as $P = 2M$ or $P = 2M+1$, and $\Omega_1 \geqslant \Omega_2 \geqslant \ldots \geqslant \Omega_N > N$. The $\mathrm{so}(P)$ and $\mathrm{u}(N)$ Hermitian irreps, $[\Xi]$ and $\{\Omega\}$, are finite dimensional, whereas the $\mathrm{sp}(2N, \mathbb{R})$ Hermitian positive discrete series irrep $\langle\Omega\rangle$ (King and Wybourne 1985) is infinite dimensional. The $\mathrm{osp}(P/2N, \mathbb{R})$ irreps will be denoted by $[\Xi\Omega\rangle$. Note that the irreps considered by Balantekin *et al* (1988, 1989) in their $\mathrm{osp}(1/2N, \mathbb{R})$ and $\mathrm{osp}(2/2N, \mathbb{R})$ CS construction correspond to the case where $\Omega_1 = \Omega_2 = \ldots = \Omega_N$. No such restriction will be made here.

Let $|[\Xi]\{\Omega\}\alpha\rangle$ denote basis states of the lowest-weight $\mathrm{so}(P) \oplus \mathrm{u}(N)$ irrep $[\Xi] \oplus \{\Omega\}$. By definition, they are annihilated by the $\mathrm{osp}(P/2N, \mathbb{R})$ lowering generators belonging to $g_{-1} \oplus g_{-2}$:

$$F^i|[\Xi]\{\Omega\}\alpha\rangle = G^{ai}|[\Xi]\{\Omega\}\alpha\rangle = J_a{}^i|[\Xi]\{\Omega\}\alpha\rangle = 0 \tag{2.19a}$$

$$D^{ij}|[\Xi]\{\Omega\}\alpha\rangle = 0. \tag{2.19b}$$

Equation (2.19b) actually follows from (A.5) and (2.19a). The lowest-weight state of $[\Xi]\oplus\{\Omega\}$ (and hence that of $[\Xi\Omega\rangle$),

$$|[\Xi]\{\Omega\}\rangle \equiv |[\Xi]\{\Omega\} lw\rangle \tag{2.20}$$

satisfies, in addition, the relations

$$\begin{aligned} &A^{ab}|[\Xi]\{\Omega\}\rangle = B^a|[\Xi]\{\Omega\}\rangle = 0 \\ &C_a{}^b|[\Xi]\{\Omega\}\rangle = 0 \qquad a>b \\ &C_a{}^a|[\Xi]\{\Omega\}\rangle = -\Xi_a|[\Xi]\{\Omega\}\rangle \\ &E_i{}^j|[\Xi]\{\Omega\}\rangle = 0 \qquad i>j \\ &E_i{}^i|[\Xi]\{\Omega\}\rangle = \Omega_{N+1-i}|[\Xi]\{\Omega\}\rangle. \end{aligned} \tag{2.21}$$

The carrier space of $[\Xi\Omega\rangle$ can be constructed from $\{|[\Xi]\{\Omega\}\alpha\rangle\}$ by applying the osp($P/2N,\mathbb{R}$) raising generators K_i, I_{ai}, $H_i{}^a$, and $D^\dagger_{ij}$ (Günaydin 1988, Günaydin and Hyun 1988). The $\mathbb{Z}$ and $\mathbb{Z}_2$ gradations of the superalgebra naturally impart similar gradations on this vector space. The lowest-weight so(P)$\oplus$u(N) subspace, spanned by $\{|[\Xi]\{\Omega\}\alpha\rangle\}$, is also the lowest $\mathbb{Z}$-grade subspace and will henceforth be referred to as the intrinsic subspace. Its $\mathbb{Z}$ grade is given by $\mathcal{N}_{\min} = \Sigma_i \Omega_i$ and its $\mathbb{Z}_2$ grade will be assumed to be $\bar{0}$.

We shall only consider here those irreps $[\Xi\Omega\rangle$ whose carrier space is a graded Hilbert space or, in other words, can be endowed with a positive semi-definite, non-degenerate, Hermitian form, denoted by a bracket $\langle\,|\,\rangle$. For physical applications, this is no essential limitation since one is mostly interested in such irreps.

3. Vector coherent states of osp($P/2N$, $\mathbb{R}$)

The osp($P/2N,\mathbb{R}$) VCS construction is based on the complex extension $g^c = g^c{}_{-2}\oplus g^c{}_{-1}\oplus g^c{}_0\oplus g^c{}_1\oplus g^c{}_2$ of decomposition (2.13) (Rowe *et al* 1988, Le Blanc and Rowe 1989, 1990, Quesne 1990b). An arbitrary vector Z belonging to $g^c{}_{-1}\oplus g^c{}_{-2}$ can be expanded as

$$Z = \tfrac{1}{2} z_{ij} D^{ij} + \theta_i F^i + \sigma_{ai} G^{ai} + \tau_i{}^a J_a{}^i \tag{3.1}$$

where the second term on the right-hand side is missing for $P = 2M$. The complex (commuting) variables $z_{ij} = z_{ji}$, $i, j = 1, \ldots, N$, and the complex (anticommuting) Grassmann variables θ_i, σ_{ai}, $\tau_i{}^a$, $a = 1, \ldots, M$, $i = 1, \ldots, N$, parametrize the complex extension of the super coset space OSp($P/2N,\mathbb{R}$)/[SO(P)$\otimes$U(N)]. The Grassmann variables are assumed to anticommute with all the odd generators and to commute with the even ones, whereas the ordinary variables commute with all the generators.

The osp($P/2N,\mathbb{R}$) VCS are then defined by

$$|z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}; \alpha\rangle = \exp(Z^*)|[\Xi]\{\Omega\}\alpha\rangle. \tag{3.2}$$

They are parametrized by the continuous variables z_{ij}, θ_i, σ_{ai}, $\tau_i{}^a$, and by the discrete index α labelling a basis of the intrinsic subspace, thence the alternative denomination of partially CS used elsewhere (Deenen and Quesne 1984a). In the special cases of osp($1/2N,\mathbb{R}$) or osp($2/2N,\mathbb{R}$), and $\Omega_1 = \ldots = \Omega_N$, the lowest-weight irrep of the stability subalgebra u(N) or so(2)$\oplus$u(N) is one dimensional, so that the VCS (3.2) reduce to the standard CS considered by Balantekin *et al* (1988, 1989).

In the general case, the VCS representation of an arbitrary state $|\Psi\rangle$, belonging to the irrep $[\Xi\Omega\rangle$ carrier space, is given by a function $\Psi(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})$ taking vector values in the intrinsic subspace. Its components

$$\Psi_\alpha(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}) = \langle z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}; \alpha|\Psi\rangle = \langle [\Xi]\{\Omega\}\alpha|\exp(Z)|\Psi\rangle \tag{3.3}$$

are holomorphic functions in the variables z_{ij} and polynomials in the Grassmann variables θ_i, σ_{ai} and $\tau_i^{\ a}$.

The carrier space of the $\mathrm{osp}(P/2N, \mathbb{R})$ VCS representation is defined as the graded Hilbert space of all such vector-valued functions which are square integrable with respect to the VCS scalar product

$$(\Psi'|\Psi)_{\mathrm{vcs}} = \sum_{\alpha\alpha'} \int [\Psi'_{\alpha'}(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})]^* \Psi_\alpha(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})\, \mathrm{d}\sigma_{\alpha'\alpha}(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}) \tag{3.4}$$

where the integration is carried out over both the ordinary and Grassmann variables (Berezin 1966) and $\mathrm{d}\sigma_{\alpha'\alpha}(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})$ is the VCS measure. The latter will not be explicitly determined in the present paper. We shall instead prove the existence of the scalar product $(\Psi'|\Psi)_{\mathrm{vcs}}$, and therefore of the corresponding measure $\mathrm{d}\sigma_{\alpha'\alpha}(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})$, by specifying an orthonormal basis with respect to this scalar product. Such a construction will be carried out by the K-matrix technique to be reviewed in the next sections.

The VCS representation $\Gamma(X)$ of an arbitrary operator X acting in the carrier space of $[\Xi\Omega\rangle$ is defined by

$$\begin{aligned}[\Gamma(X)\Psi(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})]_\alpha &= \sum_{\alpha'} \Gamma_{\alpha\alpha'}(X)\Psi_{\alpha'}(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}) \\ &= \langle [\Xi]\{\Omega\}\alpha|\exp(Z)X|\Psi\rangle.\end{aligned} \tag{3.5}$$

When X is an $\mathrm{osp}(P/2N, \mathbb{R})$ generator, $\Gamma(X)$ can be expressed as a differential operator on $\Psi(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})$, depending, in addition, on the intrinsic representation $\mathbb{A}^\dagger_{ab}$, $\mathbb{A}^{ab}$, $\mathbb{B}^\dagger_a$, $\mathbb{B}^a$, $\mathbb{C}_a^{\ b}$, and $\mathbb{E}_i^{\ j}$ of the stability subalgebra $\mathrm{so}(P)\oplus\mathrm{u}(N)$, e.g.

$$[\mathbb{A}^\dagger_{ab}\Psi(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})]_\alpha = \sum_{\alpha'} \langle [\Xi]\{\Omega\}\alpha|A^\dagger_{ab}|[\Xi]\{\Omega\}\alpha'\rangle \Psi_{\alpha'}(z, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}). \tag{3.6}$$

Using the shorthand notation

$$\nabla^{ij} = (1+\delta_{ij})\partial/\partial z_{ij} \qquad \partial^i = \partial/\partial\theta_i \qquad \partial^{ai} = \partial/\partial\sigma_{ai} \qquad \partial_a^{\ i} = \partial/\partial\tau_i^{\ a} \tag{3.7}$$

we obtain

$$\begin{aligned}
&\Gamma(D^{ij}) = \nabla^{ij} \\
&\Gamma(F^i) = \partial^i + \tfrac{1}{2}\theta_j\nabla^{ji} \qquad \Gamma(G^{ai}) = \partial^{ai} + \tfrac{1}{2}\tau_j^{\ a}\nabla^{ji} \qquad \Gamma(J_a^{\ i}) = \partial_a^{\ i} + \tfrac{1}{2}\sigma_{aj}\nabla^{ji} \\
&\Gamma(A^\dagger_{ab}) = \mathbb{A}^\dagger_{ab} + \mathscr{A}^\dagger_{ab} \qquad \Gamma(B^\dagger_a) = \mathbb{B}^\dagger_a + \mathscr{B}^\dagger_a \qquad \Gamma(C_a^{\ b}) = \mathbb{C}_a^{\ b} + \mathscr{C}_a^{\ b} \\
&\Gamma(B^a) = \mathbb{B}^a + \mathscr{B}^a \qquad \Gamma(A^{ab}) = \mathbb{A}^{ab} + \mathscr{A}^{ab} \qquad \Gamma(E_i^{\ j}) = \mathbb{E}_i^{\ j} + \mathscr{E}_i^{\ j} \\
&\Gamma(K_i) = \tau_i^{\ a}(\mathbb{B}^\dagger_a + \tfrac{1}{2}\mathscr{B}^\dagger_a) - \sigma_{ai}(\mathbb{B}^a + \tfrac{1}{2}\mathscr{B}^a) + \theta_j(\mathbb{E}_i^{\ j} + \tfrac{1}{2}\mathscr{E}_i^{\ j}) + z_{ij}\partial^j - \tfrac{1}{4}(\sigma_{ai}\tau_j^{\ a} + \tau_i^{\ a}\sigma_{aj})\theta_k\nabla^{kj} \\
&\Gamma(I_{ai}) = \tau_i^{\ b}(\mathbb{A}^\dagger_{ba} + \tfrac{1}{2}\mathscr{A}^\dagger_{ba}) - \theta_i(\mathbb{B}^\dagger_a + \tfrac{1}{2}\mathscr{B}^\dagger_a) - \sigma_{bi}(\mathbb{C}_a^{\ b} + \tfrac{1}{2}\mathscr{C}_a^{\ b}) + \sigma_{aj}(\mathbb{E}_i^{\ j} + \tfrac{1}{2}\mathscr{E}_i^{\ j}) \\
&\qquad\qquad + z_{ij}\partial_a^{\ j} - \tfrac{1}{4}(\theta_i\theta_j + \sigma_{bi}\tau_j^{\ b} + \tau_i^{\ b}\sigma_{bj})\sigma_{ak}\nabla^{kj} \\
&\Gamma(H_i^{\ a}) = \tau_i^{\ b}(\mathbb{C}_b^{\ a} + \tfrac{1}{2}\mathscr{C}_b^{\ a}) + \theta_i(\mathbb{B}^a + \tfrac{1}{2}\mathscr{B}^a) + \sigma_{bi}(\mathbb{A}^{ab} + \tfrac{1}{2}\mathscr{A}^{ab}) + \tau_j^{\ a}(\mathbb{E}_i^{\ j} + \tfrac{1}{2}\mathscr{E}_i^{\ j}) \\
&\qquad\qquad + z_{ij}\partial^{aj} - \tfrac{1}{4}(\theta_i\theta_j + \sigma_{bi}\tau_j^{\ b} + \tau_i^{\ b}\sigma_{bj})\tau_k^{\ a}\nabla^{kj}
\end{aligned} \tag{3.8}$$

$$\Gamma(D^{\dagger}_{ij}) = \tau_i{}^a\tau_j{}^b\mathbb{A}^{\dagger}_{ba} + (\theta_i\tau_j{}^a + \theta_j\tau_i{}^a)\mathbb{B}^{\dagger}_a + (\sigma_{ai}\tau_j{}^b + \sigma_{aj}\tau_i{}^b)\mathbb{C}_b{}^a + (\sigma_{ai}\theta_j + \sigma_{aj}\theta_i)\mathbb{B}^a + \sigma_{ai}\sigma_{bj}\mathbb{A}^{ba}$$
$$+ [z_{ik} + \tfrac{1}{2}(\theta_i\theta_k + \sigma_{ai}\tau_k{}^a + \tau_i{}^a\sigma_{ak})](\mathbb{E}_j{}^k + \bar{\mathscr{E}}_j{}^k)$$
$$+ [z_{jk} + \tfrac{1}{2}(\theta_j\theta_k + \sigma_{aj}\tau_k{}^a + \tau_j{}^a\sigma_{ak})](\mathbb{E}_i{}^k + \bar{\mathscr{E}}_i{}^k)$$
$$+ [z_{ik}z_{jl} - \tfrac{1}{4}(\theta_i\theta_k + \sigma_{ai}\tau_k{}^a + \tau_i{}^a\sigma_{ak})(\theta_j\theta_l + \sigma_{bj}\tau_l{}^b + \tau_j{}^b\sigma_{bl})]\nabla^{kl}$$

where

$$\mathscr{A}^{\dagger}_{ab} = \sigma_{ai}\partial_b{}^i - \sigma_{bi}\partial_a{}^i \qquad \mathscr{B}^{\dagger}_a = \sigma_{ai}\partial^i - \theta_i\partial_a{}^i \qquad \mathscr{C}_a{}^b = \sigma_{ai}\partial^{bi} - \tau_i{}^b\partial_a{}^i$$
$$\mathscr{B}^a = \theta_i\partial^{ai} - \tau_i{}^a\partial^i \qquad \mathscr{A}^{ab} = -\tau_i{}^a\partial^{bi} + \tau_i{}^b\partial^{ai} \qquad \mathscr{E}_i{}^j = z_{ik}\nabla^{kj} + \bar{\mathscr{E}}_i{}^j \qquad (3.9)$$
$$\bar{\mathscr{E}}_i{}^j = \theta_i\partial^j + \sigma_{ai}\partial^{aj} + \tau_i{}^a\partial_a{}^j.$$

Equations (3.8) and (3.9) directly result from Baker–Campbell–Hausdorff formula:

$$\exp(Z)X\exp(-Z) = X + \sum_{m=1}^{\infty}(m!)^{-1}[Z,[Z,\ldots,[Z,X]\ldots]]_m. \qquad (3.10)$$

For $X = F^i$, for instance, a straightforward application of this equation leads to the formula

$$\exp(Z)F^i = [F^i + \theta_j D^{ji}]\exp(Z). \qquad (3.11)$$

Here

$$D^{ji}\exp(Z) = \nabla^{ji}\exp(Z) \qquad (3.12)$$

and $F^i\exp(Z)$ can be found by applying (3.10) again, as follows:

$$0 = \exp(Z)\partial^i\exp(-Z)\exp(Z) = [\partial^i - F^i - \tfrac{1}{2}\theta_j D^{ji}]\exp(Z). \qquad (3.13)$$

By combining (3.11) with (3.12) and (3.13), we finally obtain the result

$$\exp(Z)F^i = [\partial^i + \tfrac{1}{2}\theta_j D^{ji}]\exp(Z) \qquad (3.14)$$

leading to the expression of $\Gamma(F^i)$ given in (3.8).

The VCS representation of the $\mathbb{Z}$-grading operator (2.17) is given by

$$\Gamma(\hat{\mathcal{N}}) = \mathcal{N}_{\min} + \hat{\mathcal{N}}_z + \hat{\mathcal{N}}_\theta + \hat{\mathcal{N}}_\sigma + \hat{\mathcal{N}}_\tau \qquad (3.15)$$

where

$$\hat{\mathcal{N}}_z = z_{ij}\nabla^{ij} \qquad \hat{\mathcal{N}}_\theta = \theta_i\partial^i \qquad \hat{\mathcal{N}}_\sigma = \sigma_{ai}\partial^{ai} \qquad \hat{\mathcal{N}}_\tau = \tau_i{}^a\partial_a{}^i \qquad (3.16)$$

are z, θ, σ and τ-number operators, respectively.

4. Orthonormal so(P)⊕u(N) vector Bargmann–Berezin basis

To apply the K-matrix technique to the osp($P/2N,\mathbb{R}$) irreps, we start by considering a vector Bargmann–Berezin (VBB) space. The latter is defined as the space of functions $\Psi(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})$ taking vector values $\Psi_\alpha(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})$ in the intrinsic subspace, and square integrable with respect to the Bargmann–Berezin (BB) scalar product

$$(\Psi'|\Psi) = \sum_\alpha \int [\Psi'_\alpha(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})]^*\Psi_\alpha(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau})\,d\mu(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\tau}) \qquad (4.1)$$

where

$$d\mu(z, \theta, \sigma, \tau) = \left(\prod_{i \leq j} d\mu[(1+\delta_{ij})^{-1/2} z_{ij}]\right)\left(\prod_{i} d\mu(\theta_i)\right)\left(\prod_{ai} d\mu(\sigma_{ai})\right)\left(\prod_{ai} d\mu(\tau_i{}^a)\right). \quad (4.2)$$

The various measures $d\mu$, appearing on the right-hand side of (4.2), are defined as Bargmann measures (Bargmann 1961)

$$d\mu(z) = \pi^{-1} \exp(-zz^*)\, dz^*\, dz \quad (4.3)$$

or Berezin measures (Berezin 1966)

$$d\mu(\theta) = \exp(-\theta\theta^*)\, d\theta^*\, d\theta \quad (4.4)$$

according to whether their argument is an ordinary complex variable z or a Grassmann variable θ. In the integration over Grassmann variables in (4.1), the normalization is fixed by

$$\int d\theta = \int d\theta^* = 0 \qquad \int \theta\, d\theta = \int d\theta^*\, \theta^* = 1. \quad (4.5)$$

Note that on the left-hand side of (4.1) (as well as on that of (3.4)), we use a round parenthesis notation to indicate that the scalar product is calculated in functional space with the BB measure (or the VCS measure). This is to be contrasted with the usual angular caret notation $\langle\Psi'|\Psi\rangle$, which represents the scalar product of two vectors belonging to the irrep carrier space.

Equations (4.3) and (4.4) define the scalar product in the standard CS representations for boson and fermion Fock states, respectively (Glauber 1963a, b, Ohnuki and Kashiwa 1978). In such representations, the boson (respectively, fermion) creation and annihilation operators are represented by z and $\partial/\partial z$ (respectively, θ and $\partial/\partial\theta$). Hence, in VBB space, the variables z_{ij}, θ_i, σ_{ai}, $\tau_i{}^a$, and the corresponding differential operators ∇^{ij}, ∂^i, ∂^{ai}, $\partial_a{}^i$ satisfy the adjoint relations

$$(z_{ij})^\dagger = \nabla^{ij} \qquad (\theta_i)^\dagger = \partial^i \qquad (\sigma_{ai})^\dagger = \partial^{ai} \qquad (\tau_i{}^a)^\dagger = \partial_a{}^i \quad (4.6)$$

with respect to the scalar product (4.1), in addition to the commutation and anticommutation relations

$$\begin{aligned} &[\nabla^{ij}, z_{kl}] = \delta_k{}^i\delta_l{}^j + \delta_k{}^j\delta_l{}^i \qquad \{\partial^i, \theta_j\} = \delta_j{}^i \\ &\{\partial^{ai}, \sigma_{bj}\} = \delta_b{}^a\delta_j{}^i \qquad \{\partial_a{}^i, \tau_j{}^b\} = \delta_a{}^b\delta_j{}^i \end{aligned} \quad (4.7)$$

with all remaining commutators or anticommutators vanishing. The set of states

$$\begin{aligned} &\left(\prod_{i\leq j} [(2n_{ij})!!]^{-1/2}(z_{ij})^{n_{ij}}\right)\left(\prod_i (\theta_i)^{n_i}\right)\left(\prod_{ai} (\sigma_{ai})^{n_{ai}}\right)\left(\prod_{ai} (\tau_i{}^a)^{n_{ia}}\right)|[\Xi]\{\Omega\}\alpha) \\ &n_{ij} = 0, 1, 2, \ldots \qquad n_i, n_{ai}, n_i{}^a = 0, 1 \end{aligned} \quad (4.8)$$

therefore form an orthonormal basis with respect to (4.1).

Starting from the intrinsic subspace, the Γ representation of the operators K_i, I_{ai}, $H_i{}^a$ and $D_{ij}^\dagger$ generates in the usual way an irreducible invariant subspace of the VBB space, which is by definition the VCS space. Although the domain of the operators $\Gamma(X)$ is restricted to the latter, we can extend it in a natural way to the whole VBB space. In the latter case, we shall speak of the extended Γ representation (Le Blanc and Rowe 1989, 1990). In the following, most equations will actually make use of this extended Γ representation, which will be denoted by the same symbol as the true VCS representation. It is, however, important to realize that, although the VCS representation is irreducible, its extension may be reducible, and even not fully reducible (see also section 5).

From (4.6), it follows that the Γ representation of $so(P)\oplus u(N)$ is compatible with the BB scalar product, which means that, for $X \in so(P)\oplus u(N)$, the adjoint $\Gamma^\dagger(X)$ of $\Gamma(X)$ with respect to this scalar product is $\Gamma(X^\dagger)$, i.e.

$$\begin{aligned} &\Gamma^\dagger(A^\dagger_{ab}) = \Gamma(A^{ab}) \qquad &&\Gamma^\dagger(B^\dagger_a) = \Gamma(B^a) \\ &\Gamma^\dagger(C_a{}^b) = \Gamma(C_b{}^a) \qquad &&\Gamma^\dagger(E_i{}^j) = \Gamma(E_j{}^i). \end{aligned} \tag{4.9}$$

In accordance with the Γ representation (3.15) of the grading operator, we define the $\mathbb{Z}$ grade of a basis state (4.8) by

$$\mathcal{N} = \mathcal{N}_{\min} + \mathcal{N}_z + \mathcal{N}_\theta + \mathcal{N}_\sigma + \mathcal{N}_\tau \tag{4.10}$$

where

$$\mathcal{N}_z = 2\sum_{i\leq j} n_{ij} \qquad \mathcal{N}_\theta = \sum_i n_i \qquad \mathcal{N}_\sigma = \sum_{ai} n_{ai} \qquad \mathcal{N}_\tau = \sum_{ai} n_i{}^a \tag{4.11}$$

are the eigenvalues of the number operators $\hat{\mathcal{N}}_z$, $\hat{\mathcal{N}}_\theta$, $\hat{\mathcal{N}}_\sigma$ and $\hat{\mathcal{N}}_\tau$, respectively. Since the $\mathbb{Z}_2$ grade of the intrinsic subspace has been chosen to be $\bar{0}$, and $\mathcal{N}_z$ is always even, consistency of the $\mathbb{Z}$- and $\mathbb{Z}_2$-gradations requires the $\mathbb{Z}_2$ grade of a basis state (4.8) to be defined by

$$\mathscr{Z} = (\mathcal{N}_\theta + \mathcal{N}_\sigma + \mathcal{N}_\tau)(\text{mod } 2). \tag{4.12}$$

Hence, the even (odd) subspace of the VBB space is spanned by the basis states with an even (odd) number of Grassmann variables.

In VBB space, it is now convenient to construct another orthonormal basis reducing the stability subalgebra $so(P)\oplus u(N)$. For such purpose, we note that the variables θ_i, σ_{ai}, $\tau_i{}^a$ and z_{ij} transform under the latter in the same way as the generators K_i, I_{ai}, $H_i{}^a$ and $D^\dagger_{ij}$ belonging to g_1 and g_2, respectively. By using (3.8), we indeed obtain the following non-vanishing commutators:

$$\begin{aligned} &[\Gamma(A^\dagger_{ab}), \tau_k{}^c] = -\delta_a{}^c\sigma_{bk} + \delta_b{}^c\sigma_{ak} \\ &[\Gamma(B^\dagger_a), \theta_k] = \sigma_{ak} \qquad [\Gamma(B^\dagger_a), \tau_k{}^c] = -\delta_a{}^c\theta_k \\ &[\Gamma(C_a{}^b), \sigma_{ck}] = \delta_c{}^b\sigma_{ak} \qquad [\Gamma(C_a{}^b), \tau_k{}^c] = -\delta_a{}^c\tau_k{}^b \\ &[\Gamma(B^a), \sigma_{ck}] = \delta_c{}^a\theta_k \qquad [\Gamma(B^a), \theta_k] = -\tau_k{}^a \\ &[\Gamma(A^{ab}), \sigma_{ck}] = \delta_c{}^a\tau_k{}^b - \delta_c{}^b\tau_k{}^a \\ &[\Gamma(E_i{}^j), \sigma_{ck}] = \delta_k{}^j\sigma_{ci} \qquad [\Gamma(E_i{}^j), \theta_k] = \delta_k{}^j\theta_i \\ &[\Gamma(E_i{}^j), \tau_k{}^c] = \delta_k{}^j\tau_i{}^c \qquad [\Gamma(E_i{}^j), z_{kl}] = \delta_k{}^j z_{il} + \delta_l{}^j z_{ik} \end{aligned} \tag{4.13}$$

which should be compared with (A.3) and (A.4).

The variables z_{ij} are therefore the components of a $[\dot{0}]\oplus\{2\dot{0}\}$ irreducible tensor $\mathfrak{z}$, whose normalization is defined by that of its highest-weight component $z_{11}/\sqrt{2}$ (here a dot over a numeral implies that this numeral is repeated as often as necessary). Whenever $P\neq 2$, the Grassmann variables σ_{ai}, θ_i and $\tau_i{}^a$ transform irreducibly according to the irrep $[1\dot{0}]\oplus\{1\dot{0}\}$, whereas, for $P=2$, σ_{1i} and $\tau_i{}^1$ are the components of two disconnected irreducible tensors, transforming according to $[1]\oplus\{1\dot{0}\}$ and $[-1]\oplus\{1\dot{0}\}$, respectively. This is an additional reason for dealing with the $P=2$ case separately in section 7. For $P\neq 2$, we denote the $[1\dot{0}]\oplus\{1\dot{0}\}$ irreducible tensor by $\mathfrak{s}$ and define its normalized highest-weight component as σ_{11} if $P\geq 3$ and θ_1 if $P=1$. As a consequence of the adjoint relations (4.6) and (4.9), the differential operators transform contragradiently to the corresponding variables. In other words, for $P\neq 2$, the operators ∇^{ij} $(\partial_a{}^i, \partial^i, \partial^{ai})$ are the components of a $[\dot{0}]\oplus\{\dot{0}-2\}$ $([1\dot{0}]\oplus\{\dot{0}-1\})$ irreducible tensor $\boldsymbol{\nabla}(\mathfrak{d})$, whose lowest-weight component is $\nabla^{11}/\sqrt{2}$ $(\partial^{11}$ if $P\geq 3$ and ∂^1 if $P=1)$.

We may construct two sets of polynomials $P_\beta^{[\dot{0}]\{\nu\}}(z)$ and $Q_\gamma^{\kappa[\lambda]\{\mu\}}(\mathfrak{s})$, transforming as the components of tensors of ranks $[\dot{0}]$ or $[\lambda]=[\lambda_1 \ldots \lambda_M]$ under so(P), and $\{\nu\}=\{\nu_1 \ldots \nu_N\}$ or $\{\mu\}=\{\mu_1 \ldots \mu_N\}$ under u(N). Here β and γ label the rows of $[\dot{0}]\oplus\{\nu\}$ and $[\lambda]\oplus\{\mu\}$, respectively, and κ distinguishes between independent Q polynomials with the same tensorial properties (no such label is needed for the P polynomials). Both sets of polynomials may be chosen orthonormal with respect to the BB scalar product (4.1):

$$(P_{\beta'}^{[\dot{0}]\{\nu'\}}|P_\beta^{[\dot{0}]\{\nu\}}) = \delta_{\{\nu'\},\{\nu\}}\delta_{\beta',\beta}$$
$$(Q_{\gamma'}^{\kappa'[\lambda']\{\mu'\}}|Q_\gamma^{\kappa[\lambda]\{\mu\}}) = \delta_{\kappa',\kappa}\delta_{[\lambda'],[\lambda]}\delta_{\{\mu'\},\{\mu\}}\delta_{\gamma',\gamma}. \tag{4.14}$$

The explicit form of the P polynomials is well known (Deenen and Quesne 1982, Le Blanc and Rowe 1987). Their highest-weight component is given by

$$P_{\rm hw}^{[\dot{0}]\{\nu\}}(z) = \mathcal{M}(\{\nu\})(z_{11})^{(\nu_1-\nu_2)/2}(z_{12,12})^{(\nu_2-\nu_3)/2}\ldots(z_{1\ldots N,1\ldots N})^{\nu_N/2} \tag{4.15}$$

where $\{\nu\}$ is a partition into non-negative even integers, $z_{1\ldots r,1\ldots r}$ denotes the determinant of order r formed from the first r rows and r columns of the $N\times N$ matrix $\|z_{ij}\|$, and $\mathcal{M}(\{\nu\})$ is the normalization coefficient

$$\mathcal{M}(\{\nu\}) = \left[\left(\prod_{i<j}^{N}(\nu_i-\nu_j+j-i)!![(\nu_i-\nu_j+j-i-1)!!]^{-1}\right)\left(\prod_{i=1}^{N}[(\nu_i+N-i)!!]^{-1}\right)\right]^{1/2}. \tag{4.16}$$

For given P and N values, the irreps $[\lambda]\oplus\{\mu\}$ characterizing the Q polynomials can be listed very easily. We indeed observe that such irreps specify the tensorial properties of the Q polynomials under the so$(P)\oplus$u(N) algebra spanned by the operators $\mathscr{A}^\dagger_{ab}$, $\mathscr{B}^\dagger_a$, $\mathscr{C}_a{}^b$, $\mathscr{B}^a$, $\mathscr{A}^{ab}$, and $\bar{\mathscr{E}}_i{}^j$, defined in (3.9). For $P>1$, the so(P) algebra is contained in a u(P) algebra, generated by

$$\begin{aligned}
&X_a{}^b = \sigma_{ai}\partial^{bi} && X_a{}^0 = \sigma_{ai}\partial^i && X_a{}^{-b} = \sigma_{ai}\partial_b{}^i\\
&X_0{}^a = \theta_i\partial^{ai} && X_0{}^0 = \theta_i\partial^i && X_0{}^{-a} = \theta_i\partial_a{}^i\\
&X_{-a}{}^b = \tau_i{}^a\partial^{bi} && X_{-a}{}^0 = \tau_i{}^a\partial^i && X_{-a}{}^{-b} = \tau_i{}^a\partial_b{}^i
\end{aligned} \tag{4.17}$$

where the operators $X_a{}^0$, $X_0{}^a$, $X_0{}^0$, $X_0{}^{-a}$ and $X_{-a}{}^0$ are only present for $P=2M+1$. Such a u(P) algebra is complementary (Moshinsky and Quesne 1970, Howe 1979) with respect to the u(N) algebra generated by $\bar{\mathscr{E}}_i{}^j$ within an antisymmetric irrep $\{1^l\dot{0}\}$ of a larger u(PN) algebra. The latter is spanned by the operators

$$\begin{aligned}
&X_{a,i}{}^{b,j} = \sigma_{ai}\partial^{bj} && X_{a,i}{}^{0,j} = \sigma_{ai}\partial^j && X_{a,i}{}^{-b,j} = \sigma_{ai}\partial_b{}^j\\
&X_{0,i}{}^{a,j} = \theta_i\partial^{aj} && X_{0,i}{}^{0,j} = \theta_i\partial^j && X_{0,i}{}^{-a,j} = \theta_i\partial_a{}^j\\
&X_{-a,i}{}^{b,j} = \tau_i{}^a\partial^{bj} && X_{-a,i}{}^{0,j} = \tau_i{}^a\partial^j && X_{-a,i}{}^{-b,j} = \tau_i{}^a\partial_b{}^j
\end{aligned} \tag{4.18}$$

where $X_{a,i}{}^{0,j}$, $X_{0,i}{}^{a,j}$, $X_{0,i}{}^{0,j}$, $X_{0,i}{}^{-a,j}$ and $X_{-a,i}{}^{0,j}$ only exist for $P=2M+1$. For any given value of l in the range $0, 1, \ldots, PN$, the allowed u(N) and u(P) irreps correspond to all conjugate partitions $\{\mu\}=\{\mu_1\ldots\mu_N\}$ and $\{\tilde{\mu}\}=\{\tilde{\mu}_1\ldots\tilde{\mu}_P\}$ of the integer l.

Hence, for $P>1$, all possible $\{\mu\}$ irreps are determined by the conditions

$$P \geqslant \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_N \geqslant 0 \qquad \sum_{i=1}^{N}\mu_i = l \tag{4.19}$$

while $[\lambda]$ runs over all the so(P) irreps contained in $\{\tilde{\mu}\}$. Repeated $[\lambda]$ irreps in a given $\{\tilde{\mu}\}$ are then distinguished by an additional label (or set of additional labels) κ.

Note that, for $N=1$, $\{\tilde{\mu}\}$ being an antisymmetric irrep, no repetition can occur so that κ is not needed. For $P=1$ and arbitrary N, only fully antisymmetric $\{\mu\}$ irreps are allowed, so that condition (4.19) is replaced by

$$\mu_1 = \mu_2 = \ldots = \mu_l = 1 \qquad \mu_{l+1} = \ldots = \mu_N = 0 \qquad l = 0, 1, \ldots, N. \tag{4.20}$$

In such a case, κ and $[\lambda]$ are missing, of course. Various examples of explicit construction of Q polynomials will be given in II.

It is now straightforward to construct an orthonormal VBB basis reducing the stability subalgebra by so(P)$\oplus$u(N) coupling of the intrinsic subspace orthonormal basis states $|[\Xi]\{\Omega\}\alpha)$ with the two sets of orthonormal polynomials in z and $\mathfrak{s}$. Such a basis is given by

$$|[\Xi\Omega\rangle\kappa[\lambda]\{\mu\}\zeta[\xi]\langle\omega\rangle\{\nu\}\rho\{h\}\chi)$$
$$= [P^{[\dot{0}]\{\nu\}}(z) \times [Q^{\kappa[\lambda]\{\mu\}}(\mathfrak{s}) \times |[\Xi]\{\Omega\})]^{\zeta[\xi]\{\omega\}}]^{\rho[\xi]\{h\}}_{\chi} \tag{4.21}$$

where the square brackets denote so(P)$\oplus$u(N) couplings, $[\xi]=[\xi_1 \ldots \xi_M]$ and $\{\omega\}=\{\omega_1 \ldots \omega_N\}$, $\{h\}=\{h_1 \ldots h_N\}$ characterize so(P) and u(N) irreps, respectively; χ labels a $[\xi]\oplus\{h\}$ basis, and ζ, ρ distinguish between repeated $[\xi]\oplus\{\omega\}$ and $[\xi]\oplus\{h\}$ irreps in $([\Xi]\oplus\{\Omega\})\otimes([\lambda]\oplus\{\mu\})$ and $([\xi]\oplus\{\omega\})\otimes([\dot{0}]\oplus\{\nu\})$, respectively. Note that in the present series of papers all couplings are assumed to be ordered sequentially from right to left and that on the left-hand side of (4.21), the u(N) irrep symbol $\{\omega\}$ has been replaced by an sp($2N$, $\mathbb{R}$) irrep symbol $\langle\omega\rangle$. The reason for this change will become clear in the next section. From (4.14) and (4.12), it results that the states (4.21) are orthonormal:

$$([\Xi\Omega\rangle\kappa'[\lambda']\{\mu'\}\zeta'[\xi']\langle\omega'\rangle\{\nu'\}\rho'\{h'\}\chi'|[\Xi\Omega\rangle\kappa[\lambda]\{\mu\}\zeta[\xi]\langle\omega\rangle\{\nu\}\rho\{h\}\chi)$$
$$= \delta_{\kappa',\kappa}\delta_{[\lambda'],[\lambda]}\delta_{\{\mu'\},\{\mu\}}\delta_{\zeta',\zeta}\delta_{[\xi'],[\xi]}\delta_{\langle\omega'\rangle,\langle\omega\rangle}\delta_{\{\nu'\},\{\nu\}}\delta_{\rho',\rho}\delta_{\{h'\},\{h\}}\delta_{\chi',\chi} \tag{4.22}$$

and that their $\mathbb{Z}_2$ grade is given by

$$\mathscr{X}(\{\omega\}) = [\sum_i (\omega_i - \Omega_i)] (\mathrm{mod}\ 2). \tag{4.23}$$

5. *K*-matrix theory of osp($P/2N$, $\mathbb{R}$)

It is obvious from (3.8) and (4.6) that the osp($P/2N$, $\mathbb{R}$) VCS representation Γ is not a star representation with respect to the BB scalar product, although it may have such a property with respect to the unknown VCS scalar product. If we were to work with the latter and the representation Γ, we would have to convert the orthonormal VBB basis, defined in (4.21), into an orthonormal VCS basis through some transformation K. It will, however, prove more convenient to keep working with the very simple BB scalar product and the orthonormal VBB basis (4.21) and instead transform the VCS representation Γ into an equivalent representation γ, satisfying the star conditions with respect to such a scalar product.

Assuming for the time being that K does not map any linear combination of VBB basis states onto the null vector so that K^{-1} is well defined, γ is then given by

$$\gamma(X) = K^{-1}\Gamma(X)K \tag{5.1}$$

and satisfies the condition

$$\gamma(X^{\dagger}) = \gamma^{\dagger}(X) \tag{5.2}$$

or

$$K^{-1}\Gamma(X^{\dagger})K = K^{\dagger}\Gamma^{\dagger}(X)(K^{-1})^{\dagger}. \tag{5.3}$$

Here X denotes any osp$(P/2N, \mathbb{R})$ generator, and $\gamma^{\dagger}$ the adjoint of γ with respect to the BB scalar product.

From (4.9), it follows that the VCS representation Γ is Hermitian on restriction to so$(P)\oplus$u(N). Hence, K may be chosen so that the VCS and VBB representations of the stability subalgebra are identical:

$$\begin{array}{lll} \Gamma(A^{\dagger}_{ab}) = \gamma(A^{\dagger}_{ab}) & \Gamma(A^{ab}) = \gamma(A^{ab}) & \Gamma(B^{\dagger}_{a}) = \gamma(B^{\dagger}_{a}) \\ \Gamma(B^{a}) = \gamma(B^{a}) & \Gamma(C_{a}{}^{b}) = \gamma(C_{a}{}^{b}) & \Gamma(E_{i}{}^{j}) = \gamma(E_{i}{}^{j}). \end{array} \tag{5.4}$$

The K matrix is therefore diagonal in the so$(P)\oplus$u(N) representation labels $[\xi]$, $\{h\}$, and independent of χ:

$$\begin{aligned} &([\Xi\Omega\rangle\kappa'[\lambda']\{\mu'\}\zeta'[\xi']\langle\omega'\rangle\{\nu'\}\rho'\{h'\}\chi'|K|[\Xi\Omega\rangle\kappa[\lambda]\{\mu\}\zeta[\xi]\langle\omega\rangle\{\nu\}\rho\{h\}\chi) \\ &\quad = \delta_{[\xi'],[\xi]}\delta_{\{h'\},\{h\}}\delta_{\chi',\chi}([\Xi\Omega\rangle\kappa'[\lambda']\{\mu'\}\zeta'[\xi]\langle\omega'\rangle\{\nu'\}\rho'\{h\}|K \\ &\quad \times|[\Xi\Omega\rangle\kappa[\lambda]\{\mu\}\zeta[\xi]\langle\omega\rangle\{\nu\}\rho\{h\}). \end{aligned} \tag{5.5}$$

Since, in physical applications, one is interested in the chain (2.12), and the construction of orthonormal sp$(2N, \mathbb{R}) \supset$ u(N) bases was extensively studied elsewhere (Rowe 1984, Rowe *et al* 1984, 1985b, Deenen and Quesne 1984b, 1985, Hecht 1987), it is convenient to require the K operator to give VCS basis states reducing (2.12), hence classified by the following labels:

$$\begin{array}{ccccccc} \text{osp}(P/2N,\mathbb{R}) & \supset & \text{so}(P)\oplus\text{sp}(2N,\mathbb{R}) & & \supset & \text{so}(P)\oplus\text{u}(N) & \\ [\Xi\Omega\rangle & \kappa[\lambda]\{\mu\}\zeta & [\xi] & \langle\omega\rangle & \{\nu\}\rho & [\xi] & \{h\} \end{array} \tag{5.6}$$

where $\langle\omega\rangle$, in particular, characterizes an sp$(2N, \mathbb{R})$ irrep. We may therefore restrict ourselves to the construction of an orthonormal VCS basis of lowest-weight so$(P)\oplus$ u(N) irrep states

$$K|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi) \tag{5.7}$$

where from now on we drop the osp$(P/2N, \mathbb{R})$ labels $[\Xi\Omega\rangle$ and denote by t the set of labels $\kappa[\lambda]\{\mu\}\zeta$, taking $T([\xi]\{\omega\})$ values.

By definition, the functions (5.7) are annihilated by the operators $\Gamma(D^{ij})$. From (3.8), it follows that they are independent of the variables z_{ij}. Hence they may be expanded into z-independent VBB basis functions

$$|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi) = [Q^{\kappa[\lambda]\{\mu\}}(\mathfrak{s}) \times |[\Xi]\{\Omega\})]^{\zeta[\xi]\{\omega\}}_{\chi}. \tag{5.8}$$

This implies that

$$(t'[\xi]\langle\omega'\rangle\{\nu'\}\rho'\{\omega\}|K|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) = \delta_{\langle\omega'\rangle,\langle\omega\rangle}\delta_{\{\nu'\},\{\dot{0}\}}(\mathscr{K}([\xi]\{\omega\}))_{t't} \tag{5.9}$$

where $\mathscr{K}([\xi]\{\omega\})$ is a submatrix of the full K matrix, whose row and column indices are $t' = \kappa'[\lambda']\{\mu'\}\zeta'$ and $t = \kappa[\lambda]\{\mu\}\zeta$, respectively. Without loss of generality, $\mathscr{K}([\xi]\{\omega\})$ may be normalized in such a way that $\mathscr{K}([\Xi]\{\Omega\}) = 1$.

From (5.3), K satisfies the following relation

$$\begin{aligned} &(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|KK^{\dagger}\Gamma^{\dagger}(\mathfrak{Z})\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\ &\quad = (t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma(\mathfrak{Z}^{\dagger})KK^{\dagger}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \end{aligned} \tag{5.10}$$

where the matrix elements are reduced with respect to the stability subalgebra so$(P)\oplus$ u(N), $\mathfrak{J}$ denotes the $[\dot{1}\dot{0}]\oplus\{\dot{0}-1\}$ irreducible tensor whose lowest-weight component is G^{11} if $P\geqslant 3$ and F^1 if $P=1$, and $\mathfrak{J}^\dagger=\pm\mathfrak{H}$ the contragradient $[\dot{1}\dot{0}]\oplus\{\dot{1}\dot{0}\}$ irreducible tensor whose highest-weight component is $G^\dagger_{11}=\pm I_{11}$ if $P\geqslant 3$ and $F^\dagger_1=\pm K_1$ if $P=1$. As in Rowe *et al* (1988), it can be easily shown that equation (5.10) leads for the product matrix $\mathcal{K}\mathcal{K}^\dagger([\xi]\{\omega\})\equiv\mathcal{K}([\xi]\{\omega\})\mathcal{K}^\dagger([\xi]\{\omega\})$ to the following recursion relation:

$$\sum_{\bar{t}'}(\mathcal{K}\mathcal{K}^\dagger([\xi']\{\omega'\}))_{t'\bar{t}'}(\bar{t}'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|[\Gamma^{(0)}(\mathfrak{J})]^\dagger\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$=\pm\sum_{\bar{t}}\{(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\mathfrak{H})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$-\sum_{t''\{\omega''\}}\frac{(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\boldsymbol{D}^\dagger)\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})}{(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{D}^\dagger)\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})}$$

$$\times(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\mathfrak{H})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\}(\mathcal{K}\mathcal{K}^\dagger([\xi]\{\omega\}))_{\bar{t}t}. \qquad (5.11)$$

Here $\boldsymbol{D}^\dagger$ denotes the $[\dot{0}]\oplus\{2\dot{0}\}$ irreducible tensor whose highest-weight component is $D^\dagger_{11}/\sqrt{2}$, $\Gamma^{(0)}(X)$ is the z- and $\boldsymbol{\nabla}$-independent component of $\Gamma(X)$, and $\Gamma^{(1)}(X)$ its component linear in z and independent of $\boldsymbol{\nabla}$.

As a consequence of the so$(P)\oplus$u(N) tensorial properties of $[\Gamma^{(0)}(\mathfrak{J})]^\dagger$, $\Gamma^{(0)}(\mathfrak{H})$ and $\Gamma^{(1)}(\mathfrak{H})$, the only non-vanishing values of their reduced matrix elements correspond to

$$[\xi']=\begin{cases}[\xi+\Delta^{(1)}(a)],[\xi-\Delta^{(1)}(a)] & \text{if } P=2M,\text{ or if } P=2M+1 \text{ and } \xi_M=0\\ [\xi+\Delta^{(1)}(a)],[\xi-\Delta^{(1)}(a)],[\xi] & \text{if } P=2M+1 \text{ and } \xi_M>0\end{cases} \qquad (5.12a)$$

and

$$\{\omega'\}=\{\omega+\Delta^{(1)}(i)\}. \qquad (5.12b)$$

Here a (respectively, i) may run over $1,\ldots,M$ (respectively, $1,\ldots,N$), and $\Delta^{(1)}(a)$ (respectively, $\Delta^{(1)}(i)$) denotes a row vector of dimension M (respectively, N) with vanishing entries everywhere except for the component a (respectively, i), which has value unity, and only standard symbols for so(P) (respectively, u(N)) irreps have to be kept on the right-hand side of (5.12a) (respectively, (5.12b)). On the other hand, the u(N) irreps $\{\omega''\}$ appearing in the summation on the right-hand side of (5.11) are given by

$$\{\omega''\}=\{\omega-\Delta^{(1)}(j)\} \qquad (5.13)$$

where j may run over $1,\ldots,N$.

By using so$(P)\oplus$u(N) tensor calculus, it will be shown in section 6 and in II that all the reduced matrix elements appearing in (5.11) can be obtained from those of the irreducible tensor $\mathfrak{s}$ introduced in section 4,

$$(\kappa'[\lambda']\{\mu'\}\|\mathfrak{s}\|\kappa[\lambda]\{\mu\})=(\kappa'[\lambda']\{\mu'\}\|Q^{[1\dot{0}]\{1\dot{0}\}}(\mathfrak{s})\|\kappa[\lambda]\{\mu\}). \qquad (5.14)$$

Hence, provided the polynomials $Q^{\kappa[\lambda]\{\mu\}}_\gamma(\mathfrak{s})$ can be constructed and all required so(P) and u(N) Racah coefficients are known, the recursion relation (5.11) can be written down in explicit form and solved by starting from the initial value $\mathcal{K}\mathcal{K}^\dagger([\Xi]\{\Omega\})=1$.

At this point, a choice can be made between the two possible signs in (2.9), (2.10) and (5.11) by imposing the requirement that $\mathcal{K}\mathcal{K}^{+}([\xi]\{\omega\})$ be a positive definite matrix for all the $so(P)\oplus u(N)$ irreps $[\xi]\oplus\{\omega\}$ encountered in the VBB basis z-independent subset (5.8) (recall that up to now K is assumed to be non-singular). In all the cases studied in II, it will turn out that the lower sign is ruled out by this condition and that, for the upper sign, some relations between the $osp(P/2N,\mathbb{R})$ irrep labels $\Xi_1,\ldots,\Xi_M,\Omega_1,\ldots,\Omega_N$ have to be satisfied whenever $P>1$. Hence, K-matrix theory provides us with a simple tool for determining whether or not a given positive discrete series irrep of $osp(P/2N,\mathbb{R})$ is equivalent to a star representation.

Restricting ourselves now to the irreps equivalent to star representations, we can determine the matrices $\mathcal{K}([\xi]\{\omega\})$ and $\mathcal{K}^{-1}([\xi]\{\omega\})$ by converting the $T\times T$ matrix $\mathcal{K}\mathcal{K}^{+}([\xi]\{\omega\})$ to diagonal form $D([\xi]\{\omega\})$ via a unitary matrix $U([\xi]\{\omega\})$:

$$U\mathcal{K}\mathcal{K}^{+}U^{+}=D=\mathrm{diag}(d_1,d_2,\ldots,d_T) \qquad (5.15)$$

with $d_r([\xi]\{\omega\})>0$ for $r=1,\ldots,T$(Hecht 1987, Hecht and Chen 1990). In (5.15), for the sake of brevity, we have dropped all the dependence on the irrep labels $[\xi]\{\omega\}$. From (5.15), we obtain

$$\mathcal{K}_{tr}=(d_r)^{1/2}U^{+}_{tr} \qquad (\mathcal{K}^{+})_{rt}=(d_r)^{1/2}U_{rt} \qquad (5.16)$$

and

$$(\mathcal{K}^{-1})_{rt}=(d_r)^{-1/2}U_{rt} \qquad ((\mathcal{K}^{+})^{-1})_{tr}=(d_r)^{-1/2}U^{+}_{tr}. \qquad (5.17)$$

The eigenstates of $\mathcal{K}\mathcal{K}^{+}([\xi]\{\omega\})$ may be labelled by index r and written as

$$|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi)=\sum_t|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi)(U^{+})_{tr}. \qquad (5.18)$$

When acting on the members of this new orthonormal set, the operators K and K^{-1} are simply given by

$$K|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi)=(d_r)^{1/2}|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi) \qquad (5.19)$$

and

$$(r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi|K^{-1}=(r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi|(d_r)^{-1/2}. \qquad (5.20)$$

It is now straightforward to calculate the $so(P)\oplus u(N)$ reduced matrix elements of the odd generators between two lowest-weight $so(P)\oplus u(N)$ irrep basis states in VBB space, i.e. by using the γ representation. Since the odd lowering generators commute with the $sp(2N,\mathbb{R})$ lowering generators D^{ij}, they can only lower states of a lowest-weight $so(P)\oplus u(N)$ irrep to other states of the same kind. Hence, taking the $so(P)\oplus u(N)$ reduced matrix elements of (5.1), for $X=\mathfrak{I}$, between two orthonormal states (5.18), we obtain

$$\begin{aligned}&(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathcal{K}([\xi']\{\omega'\})\gamma(\mathfrak{I})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\&\quad=(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\mathfrak{I})\mathcal{K}([\xi]\{\omega\})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}).\end{aligned} \qquad (5.21)$$

Combining this result with the adjoint of (6.2a), to be derived below, leads to the relation

$$\begin{aligned}&(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\mathfrak{I})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\&\quad=(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathcal{K}^{-1}([\xi']\{\omega'\})\mathfrak{d}\mathcal{K}([\xi]\{\omega\})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\&\quad=\sum_{tt'}(\mathcal{K}^{-1}([\xi']\{\omega'\}))_{r't'}(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{d}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\&\qquad\times(\mathcal{K}([\xi]\{\omega\}))_{tr}.\end{aligned} \qquad (5.22)$$

By taking the adjoint of (5.22), the reduced matrix elements of $\gamma(\mathfrak{H})$ can also be obtained as

$$
\begin{aligned}
&(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\mathfrak{H})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\
&\quad = \pm(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathscr{K}^{+}([\xi']\{\omega'\})\mathfrak{s}(\mathscr{K}^{+}([\xi]\{\omega\}))^{-1}\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\
&\quad = \pm\sum_{tt'}(\mathscr{K}^{+}([\xi']\{\omega'\}))_{r't'}(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{s}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\
&\qquad \times((\mathscr{K}^{+}([\xi]\{\omega\}))^{-1})_{tr}
\end{aligned}
\tag{5.23}
$$

where the $\pm$ sign corresponds to the $\pm$ sign in (2.9), (2.10) and (5.11).

The matrices $\mathscr{K}$, $\mathscr{K}^{+}$, $\mathscr{K}^{-1}$ and $(\mathscr{K}^{+})^{-1}$, appearing in (5.22) and (5.23), are given by (5.16) and (5.17). Hence, the matrix representation $\gamma(\mathfrak{I})$, $\gamma(\mathfrak{H})$ can finally be expressed in terms of the known reduced matrix elements of $\mathfrak{s}$, defined in (5.14), by using the relations

$$
\begin{aligned}
&(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{s}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\
&\quad = U([\Xi][\lambda][\xi'][1\dot{0}];[\xi]\zeta[\lambda']\zeta')\,U(\{\Omega\}\{\mu\}\{\omega'\}\{1\dot{0}\};\{\omega\}\zeta\{\mu'\}\zeta') \\
&\qquad \times(\kappa'[\lambda']\{\mu'\}\|\mathfrak{s}\|\kappa[\lambda]\{\mu\})
\end{aligned}
\tag{5.24}
$$

and

$$
\begin{aligned}
&(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{d}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\
&\quad = (-1)^{\psi([\xi'])+\psi([1\dot{0}])-\psi([\xi])+\varphi(\{\omega'\})+\varphi(\{1\dot{0}\})-\varphi(\{\omega\})}\left(\frac{\dim[\xi]\dim\{\omega\}}{\dim[\xi']\dim\{\omega'\}}\right)^{1/2} \\
&\qquad \times(t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\|\mathfrak{s}\|t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}).
\end{aligned}
\tag{5.25}
$$

Here the U coefficients are so(P) or u(N) Racah coefficients in unitary form (Hecht *et al* 1981, 1987), $\dim[\xi]$ and $\dim\{\omega\}$ denote the dimensions of the so(P) and u(N) irreps $[\xi]$ and $\{\omega\}$, respectively; $\varphi(\{\omega\})$ is a u(N) phase defined by

$$
\varphi(\{\omega\}) = \tfrac{1}{2}\sum_{i}(N+1-2i)\omega_i \tag{5.26}
$$

and $\psi([\xi])$ is a similar so(P) phase to be defined in II.

With the functions belonging to VBB space we can associate vectors belonging to the irrep carrier space. By replacing in the polynomials $P_{\beta}^{[\dot{0}]\{\nu\}}$ and $Q_{\gamma}^{\kappa[\lambda]\{\mu\}}$, the variables z and $\mathfrak{s}$ by the operators $\boldsymbol{D}^{+}$ and $\mathfrak{H}$, which have the same so(P)$\oplus$u(N) transformation properties, we indeed convert the VBB basis functions (4.21) and (5.8) into the state vectors

$$
\begin{aligned}
&|[\Xi\Omega\rangle\kappa[\lambda]\{\mu\}\zeta[\xi]\{\omega\}\{\nu\}\rho\{h\}\chi\rangle \\
&\quad = [P^{[\dot{0}]\{\nu\}}(\boldsymbol{D}^{+})\times[Q^{\kappa[\lambda]\{\mu\}}(\mathfrak{H})\times|[\Xi]\{\Omega\}\rangle]^{\zeta[\xi]\{\omega\}}]_{\chi}^{\rho[\xi]\{h\}}
\end{aligned}
\tag{5.27}
$$

and

$$
|t[\xi]\{\omega\}\{\dot{0}\}\{\omega\}\chi\rangle = [Q^{\kappa[\lambda]\{\mu\}}(\mathfrak{H})\times|[\Xi]\{\Omega\}\rangle]_{\chi}^{\zeta[\xi]\{\omega\}} \tag{5.28a}
$$

where we now use the usual Hilbert space notation with angular carets. Such state vectors are not characterized by a definite sp($2N,\mathbb{R}$) irrep (note that the VBB basis functions (4.21) and (5.8) have a similar property and only become characterized by a definite $\langle\omega\rangle$ after acting with the operator K). To stress this point, we use the notation $\{\omega\}$ instead of $\langle\omega\rangle$ within the kets on the left-hand side of (5.27) and (5.28a).

State vectors

$$|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi\rangle \tag{5.28b}$$

specified by a definite sp$(2N,\mathbb{R})$ irrep $\langle\omega\rangle$, and of lowest so$(P)\oplus$u(N) weight, could be constructed by combining (5.28a) with some states (5.27) with $\{\nu\}\neq\{\dot{0}\}$. Contrary to the VBB basis functions (5.8), they form a non-orthonormal set. The matrix $\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\})$ can be interpreted as their overlap matrix (Deenen and Quesne 1985, Hecht 1987, Hecht and Chen 1990):

$$\langle t'[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi\rangle=(\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\}))_{t't}. \tag{5.29}$$

The K-matrix technique main advantages consist in avoiding both the states (5.28b) painful construction and the overlap matrix (5.29) difficult evaluation, and in replacing them by the recursion relation (5.11) resolution.

A Hilbert space orthonormal basis, corresponding to the VBB orthonormal basis (5.18), is then given by

$$|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi\rangle=\sum_{t}|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi\rangle((\mathcal{K}^{\dagger}([\xi]\{\omega\}))^{-1})_{tr}. \tag{5.30}$$

Since the reduced matrix elements of $\mathfrak{Z}$ and $\mathfrak{H}$ are representation independent, they are given by

$$\langle r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{Z}\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$
$$=(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\mathfrak{Z})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \tag{5.31}$$

and

$$\langle r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{H}\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$
$$=(r'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\mathfrak{H})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}) \tag{5.32}$$

in terms of the VBB space matrix elements (5.22) and (5.23).

So far we have assumed that K does not map any linear combination of VBB basis states onto the null vector, or, in other words, that the VCS space is not a proper subspace of the VBB space. Stated differently, this means that all the states (5.28b) are linearly independent. This case occurs whenever the osp$(P/2N,\mathbb{R})$ irrep is typical, i.e. cannot be embedded into an indecomposable representation (Scheunert 1979). Such irreps exhaust all the possibilities for the osp$(1/2N,\mathbb{R})$ superalgebras. However, the remaining osp$(P/2N,\mathbb{R})$ superalgebras may also have atypical irreps, i.e. irreps which can be embedded into reducible, but not fully reducible representations. Then the VCS space is a proper subspace of the VBB space and the extended Γ representation is indecomposable although the VCS representation itself is irreducible (Le Blanc and Rowe 1989, 1990).

For atypical irreps, some of the VBB basis functions (5.8) or of the Hilbert space basis vectors (5.28b) are redundant. Hence, at least for one combination of irrep labels $[\xi]\{\omega\}$, the overlap matrix $\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\})$ of (5.29) has some zero eigenvalues, immediately signalling the presence of forbidden states. The allowed states can be designated by the index $r=1,\ldots,R$, corresponding to the non-vanishing eigenvalues d_r (Hecht 1987, Hecht and Chen 1990). Equation (5.15) now takes the form

$$U\mathscr{K}\mathscr{K}^\dagger U^\dagger = D = \mathrm{diag}(d_1, d_2, \ldots, d_R, 0, \ldots, 0) \tag{5.33}$$

where, on the right-hand side, there are $T-R$ zeros. For allowed states corresponding to $d_r \neq 0$, equations (5.16)-(5.20) remain valid, but now all matrices are rectangular and $\mathscr{K}^{-1}$ is only the left inverse of $\mathscr{K}$. These changes do not substantially alter the results derived in the first part of the present section, which may still be applied provided redundant states are consistently eliminated.

In particular, K-matrix theory remains valid so that the $T\times T$ matrices $\mathscr{K}\mathscr{K}^\dagger([\xi]\{\omega\})$ can still be obtained by solving the recursion relation (5.11) instead of having to be computed as overlap matrices. By imposing that they are positive semi-definite for all the $\mathrm{so}(P)\oplus\mathrm{u}(N)$ irreps $[\xi]\oplus\{\omega\}$ encountered in the VBB basis subset (5.8), a choice can be made between the two possible signs in (2.9), (2.10) and (5.11). This leads to the same result as before, namely that in all the examples considered in II, only the upper sign is allowed.

Once the positive semi-definite matrices $\mathscr{K}\mathscr{K}^\dagger([\xi]\{\omega\})$ are known, simple and systematic procedures are available for determining the atypicality conditions and the branching rule for the reduction of the $\mathrm{osp}(P/2N,\mathbb{R})$ irrep $[\Xi\Omega\rangle$ into irreps of its subalgebra $\mathrm{so}(P)\oplus\mathrm{sp}(2N,\mathbb{R})$, and for identifying the orthonormal basis states (5.18) and (5.30). To solve the first two problems, it is enough to determine the rank of the matrices. At least in principle, this can be done in closed, analytic form. On the contrary, to solve the third problem, numerical computation is required whenever $R>3$.

Finally, from the $\mathrm{so}(P)\oplus\mathrm{u}(N)$ reduced matrix elements of the odd generators between two lowest-weight $\mathrm{so}(P)\oplus\mathrm{u}(N)$ irrep basis states, given in (5.22), (5.23), (5.31) and (5.32), it is possible to calculate their $\mathrm{so}(P)\oplus\mathrm{sp}(2N,\mathbb{R})$ (triple) reduced matrix elements by applying the Wigner–Eckart theorem with respect to $\mathrm{sp}(2N,\mathbb{R})\supset\mathrm{u}(N)$. Since the odd generators transform under a finite-dimensional non-unitary irrep of $\mathrm{sp}(2N,\mathbb{R})$, we have to use $\mathrm{sp}(2N,\mathbb{R})$ Wigner coefficients coupling a positive discrete series unitary irrep to a non-unitary one to give another positive discrete series unitary irrep. As far as the author knows, such Wigner coefficients are known in explicit form only for $\mathrm{sp}(2,\mathbb{R})\simeq\mathrm{su}(1,1)$ (Ui 1968), so that we shall now restrict ourselves to the $\mathrm{osp}(P/2,\mathbb{R})$ superalgebras.

In such a case, index i only takes the value 1, and may be dropped. In the VBB basis states (4.21), the u(1) irreps $\{\mu\}$ and $\{\nu\}$ provide redundant labels since $\mu=\omega-\Omega$ and $\nu=h-\omega$, and hence they may also be dropped; moreover the labels κ and ρ are not necessary. The states may therefore be written as $|[\lambda]\zeta[\xi]\langle\omega\rangle\{h\}\chi\rangle$.

The $\mathrm{sp}(2,\mathbb{R})$ generators being $D^\dagger$, D and E, the corresponding su(1, 1) generators are $K_+=\frac{1}{2}D^\dagger$, $K_-=\frac{1}{2}D$ and $K_0=\frac{1}{2}E$. According to Ui (1968), an irreducible tensor T^k_q of rank k and components $q=k, k-1,\ldots,-k$ with respect to su(1, 1) is defined by the commutation relations

$$[K_0, T^k_q]=qT^k_q \qquad [K_\pm, T^k_q]=\mp[(k\mp q)(k\pm q+1)]^{1/2}T^k_{q\pm1}. \tag{5.34}$$

The operator T^k_q also transforms under the $\mathrm{sp}(2,\mathbb{R})$ irrep $\langle 2k\rangle$ and the u(1) irrep $\{2q\}$. From (A.4), it follows that the pairs of odd generators (I_a, J_a), (K, F) and (H^a, G^a) are the components $\frac{1}{2}$, $-\frac{1}{2}$ ($\{1\}$, $\{-1\}$) of an irreducible tensor of rank $\frac{1}{2}$ ($\langle 1\rangle$) with respect to su(1, 1) ($\mathrm{sp}(2,\mathbb{R})$). Hence, the $\mathrm{so}(P)\oplus\mathrm{u}(1)$ irreducible tensors $\mathfrak{H}$ and $\mathfrak{I}$ form a single $\mathrm{so}(P)\oplus\mathrm{sp}(2,\mathbb{R})$ irreducible tensor $\mathfrak{T}$, transforming under the irrep $[1\dot{0}]\oplus\langle 1\rangle$.

From a straightforward application of the Wigner–Eckart theorem, the non-vanishing $\mathrm{so}(P)\oplus\mathrm{sp}(2,\mathbb{R})$ (triple) reduced matrix elements of $\mathfrak{T}$ are given by

$$\begin{aligned}\langle[\lambda']\zeta'[\xi']\langle\omega+1\rangle|||\mathfrak{T}|||[\lambda]\zeta[\xi]\langle\omega\rangle\rangle &= ([\lambda']\zeta'[\xi']\langle\omega+1\rangle|||\gamma(\mathfrak{T})|||[\lambda]\zeta[\xi]\langle\omega\rangle)\\ &= [\langle\tfrac{1}{2}\omega\tfrac{1}{2}\omega,\tfrac{1}{2}\tfrac{1}{2}|\tfrac{1}{2}(\omega+1)\tfrac{1}{2}(\omega+1)\rangle_M]^{-1}\\ &\quad\times([\lambda']\zeta'[\xi']\langle\omega+1\rangle\{\omega+1\}\|\gamma(\mathfrak{H})\|[\lambda]\zeta[\xi]\langle\omega\rangle\{\omega\})\\ &= -[(\omega-1)/\omega]^{1/2}([\lambda']\zeta'[\xi']\langle\omega+1\rangle\{\omega+1\}\|\gamma(\mathfrak{H})\|[\lambda]\zeta[\xi]\langle\omega\rangle\{\omega\}) \qquad (5.35)\end{aligned}$$

and

$$\begin{aligned}\langle[\lambda']\zeta'[\xi']\langle\omega-1\rangle|||\mathfrak{T}|||[\lambda]\zeta[\xi]\langle\omega\rangle\rangle &= ([\lambda']\zeta'[\xi']\langle\omega-1\rangle|||\gamma(\mathfrak{T})|||[\lambda]\zeta[\xi]\langle\omega\rangle)\\ &= [\langle\tfrac{1}{2}\omega\tfrac{1}{2}\omega,\tfrac{1}{2}-\tfrac{1}{2}|\tfrac{1}{2}(\omega-1)\tfrac{1}{2}(\omega-1)\rangle_M]^{-1}\\ &\quad\times([\lambda']\zeta'[\xi']\langle\omega-1\rangle\{\omega-1\}\|\gamma(\mathfrak{J})\|[\lambda]\zeta[\xi]\langle\omega\rangle\{\omega\})\\ &= ([\lambda']\zeta'[\xi']\langle\omega-1\rangle\{\omega-1\}\|\gamma(\mathfrak{J})\|[\lambda]\zeta[\xi]\langle\omega\rangle\{\omega\}) \qquad (5.36)\end{aligned}$$

where $\langle\,,|\rangle_M$ is an su(1, 1) Wigner coefficient in the notation of Ui (1968).

6. Evaluation of so(P)⊕u(N) reduced matrix elements of $\Gamma^{(0)}(X)$ and $\Gamma^{(1)}(X)$

The power of K-matrix theory relies on the possibility of evaluating the $\mathrm{so}(P)\oplus\mathrm{u}(N)$ reduced matrix elements of the irreducible tensors $[\Gamma^{(0)}(\mathfrak{J})]^\dagger$, $\Gamma^{(0)}(\mathfrak{H})$, $\Gamma^{(1)}(\mathfrak{H})$, $\Gamma^{(0)}(\boldsymbol{D}^\dagger)$ and $\Gamma^{(1)}(\boldsymbol{D}^\dagger)$ in terms of those of $\mathfrak{s}$. In the present section, we shall outline the calculation procedure to be followed in the detailed examples of II.

From (3.8), it follows that the components of the relevant irreducible tensors are given in explicit form by

$$[\Gamma^{(0)}(G^{ai})]^\dagger=\sigma_{ai} \qquad [\Gamma^{(0)}(F^i)]^\dagger=\theta_i \qquad [\Gamma^{(0)}(J_a{}^i)]^\dagger=\tau_i{}^a \qquad (6.1a)$$

$$\begin{aligned}\Gamma^{(0)}(I_{ai})&=\tau_i{}^b(\mathbb{A}^\dagger_{ba}+\tfrac{1}{2}\mathscr{A}^\dagger_{ba})-\theta_i(\mathbb{B}^\dagger_a+\tfrac{1}{2}\mathscr{B}^\dagger_a)-\sigma_{bi}(\mathbb{C}_a{}^b+\tfrac{1}{2}\mathscr{C}_a{}^b)+\sigma_{aj}(\mathbb{E}_i{}^j+\tfrac{1}{2}\mathscr{E}_i{}^j)\\ \Gamma^{(0)}(K_i)&=\tau_i{}^a(\mathbb{B}^\dagger_a+\tfrac{1}{2}\mathscr{B}^\dagger_a)-\sigma_{ai}(\mathbb{B}^a+\tfrac{1}{2}\mathscr{B}^a)+\theta_j(\mathbb{E}_i{}^j+\tfrac{1}{2}\mathscr{E}_i{}^j), \qquad (6.1b)\\ \Gamma^{(0)}(H_i{}^a)&=\tau_i{}^b(\mathbb{C}_b{}^a+\tfrac{1}{2}\mathscr{C}_b{}^a)+\theta_i(\mathbb{B}^a+\tfrac{1}{2}\mathscr{B}^a)+\sigma_{bi}(\mathbb{A}^{ab}+\tfrac{1}{2}\mathscr{A}^{ab})+\tau_j{}^a(\mathbb{E}_i{}^j+\tfrac{1}{2}\mathscr{E}_i{}^j)\end{aligned}$$

$$\Gamma^{(1)}(I_{ai})=z_{ij}\partial_a{}^j \qquad \Gamma^{(1)}(K_i)=z_{ij}\partial^j \qquad \Gamma^{(1)}(H_i{}^a)=z_{ij}\partial^{aj} \qquad (6.1c)$$

$$\begin{aligned}\Gamma^{(0)}(D^\dagger_{ij})&=\Gamma^{(0)}_1(D^\dagger_{ij})+\Gamma^{(0)}_2(D^\dagger_{ij})\\ \Gamma^{(0)}_1(D^\dagger_{ij})&=\tau_i{}^a\tau_j{}^b\mathbb{A}^\dagger_{ba}+(\theta_i\tau_j{}^a+\theta_j\tau_i{}^a)\mathbb{B}^\dagger_a\\ &\quad+(\sigma_{ai}\tau_j{}^b+\sigma_{aj}\tau_i{}^b)\mathbb{C}_b{}^a+(\sigma_{ai}\theta_j+\sigma_{aj}\theta_i)\mathbb{B}^a+\sigma_{ai}\sigma_{bj}\mathbb{A}^{ba} \qquad (6.1d)\\ \Gamma^{(0)}_2(D^\dagger_{ij})&=\tfrac{1}{2}(\theta_i\theta_k+\sigma_{ai}\tau_k{}^a+\tau_i{}^a\sigma_{ak})(\mathbb{E}_j{}^k+\bar{\mathscr{E}}_j{}^k)+\tfrac{1}{2}(\theta_j\theta_k+\sigma_{aj}\tau_k{}^a+\tau_j{}^a\sigma_{ak})(\mathbb{E}_i{}^k+\bar{\mathscr{E}}_i{}^k)\end{aligned}$$

$$\Gamma^{(1)}(D^\dagger_{ij})=z_{ik}(\mathbb{E}_j{}^k+\bar{\mathscr{E}}_j{}^k)+z_{jk}(\mathbb{E}_i{}^k+\bar{\mathscr{E}}_i{}^k). \qquad (6.1e)$$

Equations (6.1a), (6.1b) and (6.1e) can be rewritten in tensorial form as

$$[\Gamma^{(0)}(\mathfrak{J})]^\dagger=\mathfrak{s} \qquad \Gamma^{(0)}(\mathfrak{H})=[\Lambda,\mathfrak{s}] \qquad \Gamma^{(1)}(\boldsymbol{D}^\dagger)=[\Lambda',\boldsymbol{z}] \qquad (6.2a, b, c)$$

where

$$\Lambda = \tfrac{1}{4}[-2C(\mathrm{so}(P)) + \mathscr{C}(\mathrm{so}(P)) + 2C^{(0)}(\mathrm{u}(N)) \\ - \bar{\mathscr{C}}(\mathrm{u}(N)) + (P-N-1)(\hat{\mathscr{N}}_\theta + \hat{\mathscr{N}}_\sigma + \hat{\mathscr{N}}_\tau)] \tag{6.3}$$

and

$$\Lambda' = \tfrac{1}{2}[C(\mathrm{u}(N)) - C^{(z)}(\mathrm{u}(N))] \tag{6.4}$$

are linear combinations of the operators (3.16) and of the so(P) and u(N) Casimir operators

$$C(\mathrm{so}(P)) = \tfrac{1}{2}[\gamma(A^{\dagger}_{ab})\gamma(A^{ab}) + \gamma(A^{ab})\gamma(A^{\dagger}_{ab})] \\ + \gamma(B^{\dagger}_a)\gamma(B^a) + \gamma(B^a)\gamma(B^{\dagger}_a) + \gamma(C_a{}^b)\gamma(C_b{}^a)$$

$$\mathscr{C}(\mathrm{so}(P)) = \tfrac{1}{2}(\mathscr{A}^{\dagger}_{ab}\mathscr{A}^{ab} + \mathscr{A}^{ab}\mathscr{A}^{\dagger}_{ab}) + \mathscr{B}^{\dagger}_a\mathscr{B}^a + \mathscr{B}^a\mathscr{B}^{\dagger}_a + \mathscr{C}_a{}^b\mathscr{C}_b{}^a \tag{6.5}$$

$$C^{(0)}(\mathrm{u}(N)) = \gamma^{(0)}(E_i{}^j)\gamma^{(0)}(E_j{}^i) = (\mathbb{E}_i{}^j + \bar{\mathscr{E}}_i{}^j)(\mathbb{E}_j{}^i + \bar{\mathscr{E}}_j{}^i) \qquad \bar{\mathscr{C}}(\mathrm{u}(N)) = \bar{\mathscr{E}}_i{}^j\bar{\mathscr{E}}_j{}^i$$

$$C(\mathrm{u}(N)) = \gamma(E_i{}^j)\gamma(E_j{}^i) \qquad C^{(z)}(\mathrm{u}(N)) = (\mathscr{E}_i{}^j - \bar{\mathscr{E}}_i{}^j)(\mathscr{E}_j{}^i - \bar{\mathscr{E}}_j{}^i).$$

In coupled tensor form, equation (6.1c) is

$$\Gamma^{(1)}(\mathfrak{H}) = \begin{cases} (-1)^{N-1}\sqrt{N+1}[P^{\{2\dot{0}\}}(\mathfrak{z}) \times Q^{\{\dot{0}-1\}}(\mathfrak{d})]^{\{1\dot{0}\}} & \text{if } P=1 \\ u_p(-1)^{N-1}\sqrt{N+1}\,[P^{[\dot{0}]\{2\dot{0}\}}(\mathfrak{z}) \times Q^{[1\dot{0}]\{\dot{0}-1\}}(\mathfrak{d})]^{[1\dot{0}]\{1\dot{0}\}} & \text{if } P \geqslant 3 \end{cases} \tag{6.6}$$

where $P^{\{2\dot{0}\}}(\mathfrak{z}) = P^{[\dot{0}]\{2\dot{0}\}}(\mathfrak{z}) = \mathfrak{z}$ as follows from (4.15) and (4.16). In the case where $P=1$, $Q^{\{\dot{0}-1\}}(\mathfrak{d}) = \mathfrak{d}$ is the $\{\dot{0}-1\}$ irreducible tensor which is the adjoint of $Q^{\{1\dot{0}\}}(\mathfrak{s}) = \mathfrak{s}$ and whose lowest-weight component is

$$Q^{\{\dot{0}-1\}}_{\mathrm{lw}}(\mathfrak{d}) = \partial^1. \tag{6.7}$$

In the case where $P \geqslant 3$, $Q^{[1\dot{0}]\{\dot{0}-1\}}(\mathfrak{d}) = \mathfrak{d}$ is the $[1\dot{0}]\oplus\{\dot{0}-1\}$ irreducible tensor which is the adjoint of $Q^{[1\dot{0}]\{1\dot{0}\}}(\mathfrak{s}) = \mathfrak{s}$ and whose lowest-weight component is

$$Q^{[1\dot{0}]\{\dot{0}-1\}}_{\mathrm{lw}\;\;\mathrm{lw}}(\mathfrak{d}) = \partial^{11} \tag{6.8}$$

and u_P denotes the so(P)-dependent phase factor defined by

$$Q^{[1\dot{0}]\{\dot{0}-1\}}_{\mathrm{hw}\;\;\mathrm{lw}}(\mathfrak{d}) = u_p\partial_1{}^1. \tag{6.9}$$

To prove (6.6) for $P=1$ (respectively, $P \geqslant 3$), it is enough to compare the term proportional to $z_{11}\partial^1$ (respectively, $z_{11}\partial_1{}^1$) on both sides for the highest-weight component $\Gamma^{(1)}(K_1)$ (respectively, $\Gamma^{(1)}(I_{11})$), by making use of the u(N) Wigner coefficients calculated by Biedenharn and Louck (1968). In the following, we shall use the expression of $\Gamma^{(1)}(\mathfrak{H})$ valid for $P \geqslant 3$ also in the case where $P=1$ by setting $u_1 = 1$.

As shown in (6.1d), $\Gamma^{(0)}(\boldsymbol{D}^{\dagger})$ contains two terms. The first one, which only differs from zero for $P>1$, is obtained by so(P) coupling the intrinsic so(P) generators with second-degree polynomials in $\mathfrak{s}$, transforming under the u(N) irrep $\{2\dot{0}\}$. The intrinsic generators being the components of an irreducible tensor $\mathbb{T}^{[1]\{\dot{0}\}}$ for $P=3$, of two irreducible tensors $\mathbb{T}^{[11]\{\dot{0}\}}$ and $\mathbb{T}^{[1-1]\{\dot{0}\}}$ for $P=4$, and of an irreducible tensor $\mathbb{T}^{[1^2\dot{0}]\{\dot{0}\}}$ for $P \geqslant 5$, we can write $\Gamma^{(0)}_1(\boldsymbol{D}^{\dagger})$ as

$$\Gamma^{(0)}_1(\boldsymbol{D}^{\dagger}) = \begin{cases} 0 & \text{if } P=1 \\ v_3[Q^{[1]\{2\dot{0}\}}(\mathfrak{s}) \times \mathbb{T}^{[1]\{\dot{0}\}}]^{[0]\{2\dot{0}\}} & \text{if } P=3 \\ v_4\{[Q^{[11]\{2\dot{0}\}}(\mathfrak{s}) \times \mathbb{T}^{[11]\{\dot{0}\}}]^{[\dot{0}]\{2\dot{0}\}} & \\ \quad + [Q^{[1-1]\{2\dot{0}\}}(\mathfrak{s}) \times \mathbb{T}^{[1-1]\{\dot{0}\}}]^{[\dot{0}]\{2\dot{0}\}}\} & \text{if } P=4 \\ v_P[Q^{[1^2\dot{0}]\{2\dot{0}\}}(\mathfrak{s}) \times \mathbb{T}^{[1^2\dot{0}]\{\dot{0}\}}]^{[\dot{0}]\{2\dot{0}\}} & \text{if } P \geqslant 5 \end{cases} \tag{6.10}$$

where v_P is some P-dependent coefficient. Explicit expressions for $\mathbb{T}$ and v_P will be given in II for various values of P.

The second term of $\Gamma^{(0)}(\boldsymbol{D}^\dagger)$, which only differs from zero for $N>1$, is obtained by u(N) coupling the u(N) generators $\mathbb{E}_i{}^j+\bar{\mathscr{E}}_i{}^j$ with second-degree polynomials in $\mathfrak{s}$, transforming under the u(N) irrep $\{1^2\dot{0}\}$. Apart from a scalar which does not contribute here, the former are the components of an irreducible tensor $\boldsymbol{T}^{[\dot{0}]\{1\dot{0}-1\}}$, whose highest-weight component is defined by

$$T^{[\dot{0}]\{1\dot{0}-1\}}_{\text{hw}}=-\frac{1}{\sqrt{2}}(\mathbb{E}_1{}^N+\bar{\mathscr{E}}_1{}^N). \tag{6.11}$$

The latter are the polynomials $Q^{[\dot{0}]\{1^2\dot{0}\}}(\mathfrak{s})$, whose highest-weight component is

$$Q^{[\dot{0}]\{1^2\dot{0}\}}_{\text{hw}}(\mathfrak{s})=w_P P^{-1/2}(\theta_1\theta_2+\sigma_{a1}\tau_2{}^a+\tau_1{}^a\sigma_{a2}) \tag{6.12}$$

where w_P is a P-dependent phase factor to be defined in II. Hence, $\Gamma_2^{(0)}(\boldsymbol{D}^\dagger)$ can be written as

$$\Gamma_2^{(0)}(\boldsymbol{D}^\dagger)=-w_P[P(N-1)]^{1/2}[Q^{[\dot{0}]\{1^2\dot{0}\}}(\mathfrak{s})\times\boldsymbol{T}^{[\dot{0}]\{1\dot{0}-1\}}]^{[\dot{0}]\{2\dot{0}\}}. \tag{6.13}$$

This relation can be proved by comparing the term proportional to $(\theta_1\theta_N+\sigma_{a1}\tau_N{}^a+\tau_1{}^a\sigma_{aN})(\mathbb{E}_1{}^N+\bar{\mathscr{E}}_1{}^N)$ on both sides for the highest-weight component $\Gamma_2^{(0)}(D_{11}^\dagger)/\sqrt{2}$.

From (6.2), (6.6), (6.10) and (6.13), it is now easy to express the reduced matrix elements of the various irreducible tensors in terms of those of $\mathfrak{s}$. Equation (6.2) leads directly to the results

$$\begin{aligned}(\bar{t}'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|[\Gamma^{(0)}(\mathfrak{J})]^\dagger\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\ =(\bar{t}'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{s}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\end{aligned} \tag{6.14}$$

$$\begin{aligned}(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\mathfrak{H})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\ =[\Lambda([\lambda']\{\mu'\},[\xi']\{\omega'\})-\Lambda([\bar{\lambda}]\{\bar{\mu}\},[\xi]\{\omega\})]\\ \times(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{s}\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{D}^\dagger)\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})\\ =[\Lambda'(\{2\dot{0}\},\{\omega'\})-\Lambda'(\{\dot{0}\},\{\omega''\})]\\ \times(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|z\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})\end{aligned} \tag{6.16}$$

where

$$\begin{aligned}\Lambda([\lambda]\{\mu\},[\xi]\{\omega\})\\ =\tfrac{1}{4}\Big[-2\sum_a\xi_a(\xi_a+P-2a)+\sum_a\lambda_a(\lambda_a+P-2a)\\ +2\sum_i\omega_i(\omega_i+N-2i+1)-\sum_i\mu_i(\mu_i+2N-P-2i+2)\Big]\end{aligned} \tag{6.17}$$

and

$$\Lambda'(\{\nu\},\{\omega\})=\tfrac{1}{2}\sum_i[\omega_i(\omega_i+N-2i+1)-\nu_i(\nu_i+N-2i+1)] \tag{6.18}$$

are the eigenvalues, corresponding to the state (4.21), of the operators Λ and Λ', defined in (6.3) and (6.4), respectively. When use is made of (5.12*b*), (5.13), and of the relation (Le Blanc and Rowe 1987)

$$(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|z\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}) = (\{2\dot{0}\}\|z\|\{\dot{0}\}) = 1 \tag{6.19}$$

equation (6.16) takes the much simpler form

$$(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{D}^{\dagger})\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}) = \omega_i + \omega_j - i - j. \tag{6.20}$$

By applying the Wigner–Eckart theorem with respect to so(P)$\oplus$u(N), the reduced matrix element of $\Gamma^{(1)}(\mathfrak{H})$, as given in (6.6), can be written as

$$\begin{aligned}(t''[\xi']\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\mathfrak{H})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\\ = u_P(-1)^{N-1}\sqrt{N+1}\, U(\{\omega\}\{\dot{0}-1\}\{\omega'\}\{2\dot{0}\};\{\omega''\}\{1\dot{0}\})\\ \times(t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}\|\mathfrak{d}\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\end{aligned} \tag{6.21}$$

where, on the right-hand side, the u(N) Racah coefficient can be calculated from (A.9) of Le Blanc and Hecht (1987) by using some symmetry properties of Racah coefficients (Hecht *et al* 1981), and the reduced matrix element of $\mathfrak{d}$ is given in (5.25).

By proceeding in the same way, the reduced matrix element of $\Gamma_2^{(0)}(\boldsymbol{D}^{\dagger})$, defined in (6.13), can be expressed as

$$\begin{aligned}(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma_2^{(0)}(\boldsymbol{D}^{\dagger})\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})\\ = -w_P[P(N-1)]^{1/2}U(\{1^2\dot{0}\}\{1\dot{0}-1\}\{\omega'\}\{\omega''\};\{2\dot{0}\}\{\omega''\}(\rho=1))\\ \times(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|Q^{[\dot{0}]\{1^2\dot{0}\}}(\mathfrak{s})\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})\\ \times(t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}\|\boldsymbol{T}^{[\dot{0}]\{1\dot{0}-1\}}\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}).\end{aligned} \tag{6.22}$$

Here the reduced matrix element of $\boldsymbol{T}^{[\dot{0}]\{1\dot{0}-1\}}$ is given by (Louck and Biedenharn 1970)

$$(t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}\|\boldsymbol{T}^{[\dot{0}]\{1\dot{0}-1\}}\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}) = [\tfrac{1}{2}g(\{\omega''\})]^{1/2} \tag{6.23}$$

where

$$g(\{\omega\}) = N^{-1}\sum_{i<j}(\omega_i-\omega_j)(\omega_i-\omega_j+2j-2i). \tag{6.24}$$

The u(N) Racah coefficient, where $\rho=1$ refers to the case where the $\{1\dot{0}^{-1}\}$ irreducible tensor is made of the set of su(N) generators, has been calculated elsewhere (Quesne 1990c) and is equal to

$$\begin{aligned}U(\{1^2\dot{0}\}\{1\dot{0}-1\}\{\omega'\}\{\omega''\};\{2\dot{0}\}\{\omega''\}\{\rho=1)\}\\ = [2(N-1)g(\{\omega''\})]^{-1/2}[(\omega_i-\omega_j+j-i)(\omega_i-\omega_j+j-i+2)]^{1/2}\end{aligned} \tag{6.25}$$

where $\{\omega'\}$ and $\{\omega''\}$ are given by (5.12*b*) and (5.13), respectively. Finally, by applying the Wigner–Eckart theorem with respect to u(N), the reduced matrix element of $Q^{[\dot{0}]\{1^2\dot{0}\}}(\mathfrak{s})$ can be written as

$$\begin{aligned}(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|Q^{[\dot{0}]\{1^2\dot{0}\}}(\mathfrak{s})\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})\\ = U(\{\Omega\}\{\mu''\}\{\omega'\}\{1^2\dot{0}\};\{\omega''\}\zeta''\{\mu'\}\zeta')(\kappa'[\lambda']\{\mu'\}\|Q^{[\dot{0}]\{1^2\dot{0}\}}(\mathfrak{s})\|\kappa''[\lambda'']\{\mu''\})\end{aligned} \tag{6.26}$$

where the reduced matrix element on the right-hand side is expressible in terms of (5.14), and of so(P) and u(N) Racah coefficients. In II, detailed expressions will be given for the reduced matrix element of $\Gamma_1^{(0)}(\boldsymbol{D}^+)$ for various P values.

7. The case of osp(2/2N, ℝ)

Whenever $P=2$, there are essentially two modifications with respect to the general theory developed for the cases where $P\neq 2$ in the previous sections: on one hand, osp(2/2N, ℝ) may have grade star irreps, and on the other hand its odd raising (or lowering) generators do not transform irreducibly under the stability subalgebra so(2)⊕ u(N). In the present section, we shall successively review the consequences of these two differences by starting with the latter.

Since index a takes the single value 1, it may be dropped, so that the so(2) generator is denoted by C, and the odd raising (lowering) generators by I_i and H_i (G^i and J^i). From (A.3) and (A.4), it is clear that the operators I_i and H_i (G^i and J^i) are the components of two separate irreducible tensors $\boldsymbol{I}$ and $\boldsymbol{H}$ ($\boldsymbol{G}$ and $\boldsymbol{J}$), transforming under the so(2)⊕u(N) irreps $[1]\oplus\{1\dot{0}\}$ and $[-1]\oplus\{1\dot{0}\}$ ($[-1]\oplus\{\dot{0}-1\}$ and $[1]\oplus\{\dot{0}-1\}$). In the $N=1$ case, the odd generators form two separate so(2)⊕sp(2, ℝ) irreducible tensors $\mathscr{I}=(I, J)$, and $\mathscr{H}=(H, G)$ transforming under the irreps $[1]\oplus\langle 1\rangle$ and $[-1]\oplus\langle 1\rangle$, respectively. The same holds true for the Grassmann variables σ_i and τ_i (and their corresponding differential operators $\partial/\partial\sigma_i$ and $\partial/\partial\tau_i$), which are the components of two so(2)⊕u(N) irreducible tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ ($\boldsymbol{\partial/\partial\sigma}$ and $\boldsymbol{\partial/\partial\tau}$). Such pairs of irreducible tensors replace the single irreducible tensors $\mathfrak{H}$, $\mathfrak{I}$, $\mathfrak{T}$, $\mathfrak{s}$ and $\mathfrak{d}$, respectively.

As a consequence, the Q polynomials, used in constructing the orthonormal VBB basis (4.21) reducing the stability subalgebra so(2)⊕u(N), are functions of the two irreducible tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. They may be denoted by $Q^{[\lambda]\{\mu\}}_{\gamma}(\boldsymbol{\sigma}, \boldsymbol{\tau})$, where the additional label κ is not needed since no so(2) irrep $[\lambda]$ is repeated within a given u(2) irrep $\{\tilde{\mu}\}$. Both equations (5.24) and (5.25) now split up into a set of two equations, wherein all so(P)-dependent factors disappear.

Let us now review the star and grade star irreps of osp(2/2N, ℝ). For such superalgebras, the adjoint operation in the subalgebra so(2)⊕sp(2N, ℝ) can be extended not only to the adjoint operation (2.9), (2.10), but also to the grade adjoint operation (Scheunert *et al* 1977)

$$(I_i)^{\ddagger}=\pm G^i \qquad (G^i)^{\ddagger}=\mp I_i \qquad (H_i)^{\ddagger}=\mp J^i \qquad (J^i)^{\ddagger}=\pm H_i. \tag{7.1}$$

Hence, in addition to γ representations fulfilling the star conditions (5.2), there may also exist γ representations satisfying the grade star conditions

$$\gamma(X^{\ddagger})=\gamma^{\ddagger}(X) \tag{7.2}$$

where X denotes any osp(2/2N, ℝ) generator. Their matrix elements in VBB space are such that

$$(x|\gamma(X^{\dagger})|y)=(y|\gamma(X)|x)^* \tag{7.3}$$

or

$$(x|\gamma(X^{\ddagger})|y)=(-1)^{\mathscr{Z}(x)\mathscr{Z}(X)}(y|\gamma(X)|x)^* \tag{7.4}$$

respectively. Here $|x)$ and $|y)$ are any two VBB basis states (4.21), $\mathscr{Z}(x)$ the $\mathbb{Z}_2$ grade of $|x)$ defined in (4.23), and $\mathscr{Z}(X)$ the $\mathbb{Z}_2$ grade of generator X.

For both types of irreps, we are then led to replace the recursion relation (5.11) by a pair of recursion relations, which can be written in a unified way as

$$\sum_{\bar{t}'} (\mathcal{K}\mathcal{K}^{\dagger}([\xi+1]\{\omega'\}))_{t'\bar{t}'}(\bar{t}'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|[\Gamma^{(0)}(\boldsymbol{G})]^{\dagger}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$= \delta_{+}\sum_{\bar{t}}\{(t'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\boldsymbol{I})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$-\sum_{t''\{\omega''\}}\frac{(t'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\boldsymbol{D}^{\dagger})\|t''[\xi+1]\langle\omega''\rangle\{\dot{0}\}\{\omega''\})}{(t''[\xi+1]\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{D}^{\dagger})\|t''[\xi+1]\langle\omega''\rangle\{\dot{0}\}\{\omega''\})}$$

$$\times(t''[\xi+1]\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{I})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\}(\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\}))_{\bar{t}t} \tag{7.5}$$

and

$$\sum_{\bar{t}'} (\mathcal{K}\mathcal{K}^{\dagger}([\xi-1]\{\omega'\}))_{t'\bar{t}'}(\bar{t}'[\xi-1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|[\Gamma^{(0)}(\boldsymbol{J})]^{\dagger}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$= \delta_{-}\sum_{\bar{t}}\{(t'[\xi-1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\boldsymbol{H})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$-\sum_{t''\{\omega''\}}\frac{(t'[\xi-1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(\boldsymbol{D}^{\dagger})\|t''[\xi-1]\langle\omega''\rangle\{\dot{0}\}\{\omega''\})}{(t''[\xi-1]\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{D}^{\dagger})\|t''[\xi-1]\langle\omega''\rangle\{\dot{0}\}\{\omega''\})}$$

$$\times(t''[\xi-1]\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\boldsymbol{H})\|\bar{t}[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})\}(\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\}))_{\bar{t}t} \tag{7.6}$$

where

$$\delta_{+} = \delta_{-} = \pm 1 \tag{7.7}$$

or

$$\delta_{+} = -\delta_{-} = \mp(-1)^{\mathcal{F}(\{\omega'\})} \tag{7.8}$$

according to whether one considers a star or grade star irrep. The conditions of existence for star and grade star irreps and the branching rule for their reduction into a direct sum of so(2)$\oplus$sp(2N, $\mathbb{R}$) irreps can then be determined in a parallel way.

The u(N) reduced matrix elements of the odd lowering generators between two lowest-weight so(2)$\oplus$u(N) irrep basis states are given by relations analogous to (5.22), namely

$$(r'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\boldsymbol{J})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$=\sum_{tt'}(\mathcal{K}^{-1}([\xi+1]\{\omega'\}))_{r't'}(t'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\boldsymbol{\partial/\partial\tau}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$\times(\mathcal{K}([\xi]\{\omega\}))_{tr} \tag{7.9}$$

and a similar relation for $\gamma(\boldsymbol{G})$ with $[\xi+1]$ and $\boldsymbol{\partial/\partial\tau}$ replaced by $[\xi-1]$ and $\boldsymbol{\partial/\partial\sigma}$ respectively. Those of the odd raising generators can be obtained from them by using either the adjoint relations (2.9) and (7.3) or the grade adjoint relations (7.1) and (7.4). They can be written as

$$(r'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\boldsymbol{I})\|r[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$=\delta_{+}\sum_{tt'}(\mathcal{K}^{\dagger}([\xi+1]\{\omega'\}))_{r't'}(t'[\xi+1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\boldsymbol{\sigma}\|t[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$

$$\times((\mathcal{K}^{\dagger}([\xi]\{\omega\}))^{-1})_{tr} \tag{7.10}$$

and a similar relation for $\gamma(\boldsymbol{H})$ with $[\xi+1]$, δ_{+} and $\boldsymbol{\sigma}$ replaced by $[\xi-1]$, δ_{-} and $\boldsymbol{\tau}$, respectively.

Finally, the tensorial form of the operators $\Gamma^{(0)}(X)$ and $\Gamma^{(1)}(X)$, given in (6.2*a*), (6.2*b*), (6.6), (6.10) and (6.13) for $P \neq 2$, now becomes

$$[\Gamma^{(0)}(\boldsymbol{G})]^{\dagger} = \boldsymbol{\sigma} \qquad [\Gamma^{(0)}(\boldsymbol{J})]^{\dagger} = \boldsymbol{\tau} \tag{7.11}$$

$$\Gamma^{(0)}(\boldsymbol{I}) = [\Lambda, \boldsymbol{\sigma}] \qquad \Gamma^{(0)}(\boldsymbol{H}) = [\Lambda, \boldsymbol{\tau}] \tag{7.12}$$

$$\Gamma^{(1)}(\boldsymbol{I}) = (-1)^{N-1}\sqrt{N+1}\,[P^{[0]\{2\dot{0}\}}(z) \times Q^{[1]\{\dot{0}-1\}}(\boldsymbol{\partial}/\boldsymbol{\partial\tau})]^{[1]\{10\}} \tag{7.13a}$$

$$\Gamma^{(1)}(\boldsymbol{H}) = (-1)^{N-1}\sqrt{N+1}\,[P^{[0]\{2\dot{0}\}}(z) \times Q^{[-1]\{\dot{0}-1\}}(\boldsymbol{\partial}/\boldsymbol{\partial\sigma})]^{[-1]\{1\dot{0}\}} \tag{7.13b}$$

$$\Gamma_1^{(0)}(\boldsymbol{D}^{\dagger}) = v_2 \Xi Q^{[0]\{2\dot{0}\}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \tag{7.14}$$

$$\Gamma_2^{(0)}(\boldsymbol{D}^{\dagger}) = -w_2[2(N-1)]^{1/2}[Q^{[0]\{1^2\dot{0}\}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \times \boldsymbol{T}^{[0]\{1\dot{0}-1\}}]^{[0]\{2\dot{0}\}} \tag{7.15}$$

while that of $\Gamma^{(1)}(\boldsymbol{D}^{\dagger})$, given in (6.2*c*) for $P \neq 2$, remains unchanged. In (7.13), the irreducible tensors $Q^{[1]\{\dot{0}-1\}}_{\gamma}(\boldsymbol{\partial}/\boldsymbol{\partial\tau})$ and $Q^{[-1]\{\dot{0}-1\}}_{\gamma}(\boldsymbol{\partial}/\boldsymbol{\partial\sigma})$ are defined in such a way that their lowest-weight components are

$$Q^{[1]\{\dot{0}-1\}}_{\text{lw}}(\boldsymbol{\partial}/\boldsymbol{\partial\tau}) = \partial/\partial\tau_1 \qquad Q^{[-1]\{\dot{0}-1\}}_{\text{lw}}(\boldsymbol{\partial}/\boldsymbol{\partial\sigma}) = \partial/\partial\sigma_1 \tag{7.16}$$

respectively. In deriving (7.14), we used the fact that the single intrinsic so(2) generator $\mathbb{C}$ may be replaced by its eigenvalue Ξ. As a consequence of (7.11)-(7.13) and (7.15), each of the equations (6.14), (6.15) and (6.21) (where $u_2 = 1$) splits up into a set of two equations, while equations (6.16) and (6.22) are still valid.

Appendix. Supercommutation relations of osp($P/2N$, ℝ)

In the present appendix, we list the non-vanishing commutators and anticommutators of the osp($2M+1/2N$, ℝ) generators. Those of the osp($2M/2N$, ℝ) generators can be obtained from them by dropping the relations containing the operators $B_a^{\dagger}$, B^a, K_i and F^i.

The commutators of the even generators, i.e. the generators of the so($2M+1$)⊕ sp($2N$, ℝ) subalgebra, are given by the relations

$$\begin{aligned}
&[C_a{}^b, C_c{}^d] = \delta_c{}^b C_a{}^d - \delta_a{}^d C_c{}^b \\
&[C_a{}^b, A^{\dagger}_{cd}] = \delta_c{}^b A^{\dagger}_{ad} - \delta_d{}^b A^{\dagger}_{ac} \qquad [C_a{}^b, B^{\dagger}_c] = \delta_c{}^b B^{\dagger}_a \\
&[A^{ab}, A^{\dagger}_{cd}] = -\delta_c{}^a C_d{}^b + \delta_c{}^b C_d{}^a + \delta_d{}^a C_c{}^b - \delta_d{}^b C_c{}^a \\
&[B^a, A^{\dagger}_{bc}] = \delta_c{}^a B^{\dagger}_b - \delta_b{}^a B^{\dagger}_c \\
&[B^{\dagger}_a, B^{\dagger}_b] = -A^{\dagger}_{ab} \qquad [B^a, B^{\dagger}_b] = -C_b{}^a
\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}
&[E_i{}^j, E_k{}^l] = \delta_k{}^j E_i{}^l - \delta_i{}^l E_k{}^j \qquad [E_i{}^j, D^{\dagger}_{kl}] = \delta_k{}^j D^{\dagger}_{il} + \delta_l{}^j D^{\dagger}_{ik} \\
&[D^{ij}, D^{\dagger}_{kl}] = \delta_k{}^i E_l{}^j + \delta_k{}^j E_l{}^i + \delta_l{}^i E_k{}^j + \delta_l{}^j E_k{}^i
\end{aligned} \tag{A.2}$$

as well as by those which can be derived from them by using the adjoint conditions (2.6) and (2.7).

The commutators of the even generators with the odd ones are

$$\begin{aligned}
&[C_a{}^b, G^{ci}] = -\delta_a{}^c G^{bi} \qquad [C_a{}^b, H_i{}^c] = -\delta_a{}^c H_i{}^b \\
&[C_a{}^b, I_{ci}] = \delta_c{}^b I_{ai} \qquad [C_a{}^b, J_c{}^i] = \delta_c{}^b J_a{}^i \\
&[A^\dagger_{ab}, G^{ci}] = -\delta_a{}^c J_b{}^i + \delta_a{}^b J_c{}^i \qquad [A^\dagger_{ab}, H_i{}^c] = -\delta_a{}^c I_{bi} + \delta_b{}^c I_{ai} \\
&[B^\dagger_a, F^i] = J_a{}^i \qquad [B^\dagger_a, G^{bi}] = -\delta_a{}^b F^i \\
&[B^\dagger_a, H_i{}^b] = -\delta_a{}^b K_i \qquad [B^\dagger_a, K_i] = I_{ai} \\
&[B^a, F^i] = -G^{ai} \qquad [B^a, I_{bi}] = \delta_b{}^a K_i \\
&[B^a, J_b{}^i] = \delta_b{}^a F^i \qquad [B^a, K_i] = -H_i{}^a \\
&[A^{ab}, I_{ci}] = \delta_c{}^a H_i{}^b - \delta_c{}^b H_i{}^a \qquad [A^{ab}, J_c{}^i] = \delta_c{}^a G^{bi} - \delta_c{}^b G^{ai}
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
&[E_i{}^j, F^k] = -\delta_i{}^k F^j \qquad [E_i{}^j, G^{ak}] = -\delta_i{}^k G^{aj} \qquad [E_i{}^j, H_k{}^a] = \delta_k{}^j H_i{}^a \\
&[E_i{}^j, I_{ak}] = \delta_k{}^j I_{ai} \qquad [E_i{}^j, J_a{}^k] = -\delta_i{}^k J_a{}^j \qquad [E_i{}^j, K_k] = \delta_k{}^j K_i \\
&[D^\dagger_{ij}, F^k] = -\delta_i{}^k K_j - \delta_j{}^k K_i \qquad [D^\dagger_{ij}, G^{ak}] = -\delta_i{}^k H_j{}^a - \delta_j{}^k H_i{}^a \\
&[D^\dagger_{ij}, J_a{}^k] = -\delta_i{}^k I_{aj} - \delta_j{}^k I_{ai} \qquad [D^{ij}, H_k{}^a] = \delta_k{}^i G^{aj} + \delta_k{}^j G^{ai} \\
&[D^{ij}, I_{ak}] = \delta_k{}^i J_a{}^j + \delta_k{}^j J_a{}^i \qquad [D^{ij}, K_k] = \delta_k{}^i F^j + \delta_k{}^j F^i.
\end{aligned} \tag{A.4}$$

Finally, the anticommutators of the odd generators are given by

$$\begin{aligned}
&\{F^i, F^j\} = D^{ij} \qquad \{F^i, H_j{}^a\} = \delta_j{}^i B^a \qquad \{F^i, I_{aj}\} = -\delta_j{}^i B^\dagger_a \qquad \{F^i, K_j\} = E_j{}^i \\
&\{G^{ai}, H_j{}^b\} = -\delta_j{}^i A^{ab} \qquad \{G^{ai}, I_{bj}\} = \delta_b{}^a E_j{}^i - \delta_j{}^i C_b{}^a \\
&\{G^{ai}, J_b{}^j\} = \delta_b{}^a D^{ij} \qquad \{G^{ai}, K_j\} = -\delta_j{}^i B^a \\
&\{H_i{}^a, I_{bj}\} = \delta_b{}^a D^\dagger_{ij} \qquad \{H_i{}^a, J_b{}^j\} = \delta_b{}^a E_i{}^j + \delta_i{}^j C_b{}^a \\
&\{I_{ai}, J_b{}^j\} = -\delta_i{}^j A^\dagger_{ab} \qquad \{J_a{}^i, K_j\} = \delta_j{}^i B^\dagger_a \qquad \{K_i, K_j\} = D^\dagger_{ij}.
\end{aligned} \tag{A.5}$$

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